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A NUMERICAL MODEL OF THE GENERAL CIRCULATION OF THE ATMOSPHERE OVER A HEMISPHERE: A Translation from *Trudy, Main Geophysical Observatory*, Leningrad, USSR, Vol. 256 (1970), pp. 3-44, by M. E. Shvets, L. R. Dmitrieva-Arrago, L. N. Magazenkov, L. V. Samoilova, and B. E. Shneerov

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EDITORS' PREFACE

This translation has been prepared for the information of the atmospheric research community engaged in the application of general circulation models to the numerical simulation of climate. The model described herein has a number of features in common with other such models now in use. The parameterizations of short- and long-wave radiation transfer, the treatments of cloudiness and of convection, and the calculation of the fluxes in the surface boundary layer are of particular interest. Although no actual numerical solutions are given in this paper, the use of the so-called splitting method of Marchuk is envisaged. These and other features may make the paper of value to those engaged in similar modeling efforts.

To the editors' knowledge, this paper represents the most comprehensive formulation of an atmospheric general circulation model undertaken in the Soviet Union as of its publication date (1970). Although the model is discussed here as a hemispheric one, the extension to a global domain is straightforward.

SUMMARY

Numerical experiments, based on the physical laws of atmospheric motion and the processes of heat and moisture exchange, have attained importance for developing numerical methods of weather forecasting and for studying the genesis of climate and the general circulation. Achievements in this field[1,2] unquestionably occupy an important place in the theory of forecasting and general atmospheric circulation.

However, there is a basic inadequacy in the indicated studies, due to the fact that climatological values of moisture and cloudiness are used in calculations of radiational heat fluxes. As a result, the dynamic characteristics and heat fluxes are not necessarily in balance.

In this article is presented a scheme of the general atmospheric circulation that differs from the preceeding studies of the problem in physical content, since cloudiness and moisture are not given, but are determined in the process of integrating the appropriate equations. However, the ocean temperature is considered to be known from climatological data. To reject assignment of this temperature would require that a system of equations describing the laws of oceanic circulation be included in the model. Consideration of such a complete problem is premature at present for a number of reasons.

The general circulation in the northern hemisphere for the cold period of the year is considered in this study. As a result, it is possible to disregard evaporation from dry land, which greatly facilitates solution of the problem.

SYMBOLS

The following basic symbols are used in the article.

A = the flux of longwave radiation, downward
a = the mean radius of the earth
B = the flux of longwave radiation, upward
C = the amount of cloudiness
c_p = specific heat of air at constant pressure
c_v = specific heat of air at constant volume
α = radiating capacity of underlying surface
F = radiation balance
f = relative humidity
g = acceleration of gravity
H = altitude of σ surface
H_s = altitude of the earth's surface
L = latent heat of condensation
$l = 2\Omega \cos \theta$ = Coriolis parameter
p = pressure
p_s = pressure at the earth's surface
Π_T = turbulent vertical heat flux
Π_q = turbulent vertical moisture flux
Q = flux of total radiation
q = specific humidity
r = albedo of underlying surface
R_B = gas constant of the air
S = flux of direct solar radiation
T = air temperature
T_s = temperature of the earth's surface
t = time
u = component of wind velocity along latitudinal circumference with positive direction to the east
v = component of wind velocity along meridian with positive direction to the South

λ = longitude

γ = coefficient of horizontal macroturbulent diffusion [also, scattering angle, Ed.]

ρ = air density

$\sigma = \frac{p}{p_s}$ = coordinate along a vertical

τ_λ = component of frictional stress along latitudinal circumference

τ_θ = component of frictional stress along meridian

θ = co-latitude

Ω = angular velocity of earth's rotation

$\omega = \frac{dp}{dt}$ = vertical-p velocity

$\bar{\omega} = \frac{d\sigma}{dt}$ = vertical-σ velocity

$\kappa = \frac{c_p}{c_v}$ = [also, scattering function, Ed.]

[Editors' note: There are also numerous other symbols defined locally in the text.]

CONTENTS

I. BASIC EQUATIONS AND BOUNDARY CONDITIONS OF THE PROBLEM AND METHOD OF SOLUTION

1. SYSTEM OF EQUATIONS

Consider a system of thermohydrodynamic equations describing the transfer of momentum, heat, and moisture in the atmosphere. Let a global system of coordinates be used; the relationship of pressure, p, to its value at the earth's surface, p_s, is given as an independent variable along the vertical (the so-called σ-system of coordinates).(3) The system of equations includes:

1. Equation of motion

$$\frac{\partial u}{\partial t} + \frac{u}{a \sin\theta}\frac{\partial u}{\partial \lambda} + \frac{v}{a}\frac{\partial u}{\partial \theta} + \bar{\omega}\frac{\partial u}{\partial \sigma} + \frac{uv \operatorname{ctg}\theta}{a} + lv$$

$$= -\frac{g}{a \sin\theta}\frac{\partial H}{\partial \lambda} - \frac{R_B T}{p_s a \sin\theta}\frac{\partial p_s}{\partial \lambda} + \nu\nabla^2 u - \frac{g}{p_s}\frac{\partial \tau_\lambda}{\partial \sigma}, \tag{1.1}$$

$$\frac{\partial v}{\partial t} + \frac{u}{a \sin\theta}\frac{\partial v}{\partial \lambda} + \frac{v}{a}\frac{\partial v}{\partial \theta} + \bar{\omega}\frac{\partial v}{\partial \sigma} - \frac{u^2 \operatorname{ctg}\theta}{a} - lu$$

$$= -\frac{g}{a}\frac{\partial H}{\partial \theta} - \frac{R_B T}{p_s a}\frac{\partial p_s}{\partial \theta} + \nu\nabla^2 v - \frac{g}{p_s}\frac{\partial \tau_\theta}{\partial \sigma}, \tag{1.2}$$

$$\nabla^2 = \frac{1}{a^2}\left[\frac{\partial^2}{\partial \theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \lambda^2}\right];$$

2. Equation of heat flux

$$\frac{\partial T}{\partial t} + \frac{u}{a \sin\theta}\frac{\partial T}{\partial \lambda} + \frac{v}{a}\frac{\partial T}{\partial \theta} + \bar{\omega}\frac{\partial T}{\partial \sigma} - \frac{\kappa - 1}{\kappa}\frac{T\omega}{p_s \sigma}$$

$$= \nu\nabla^2 T + \frac{g}{p_s c_p}\frac{\partial}{\partial \sigma}(\Pi_T + F); \tag{1.3}$$

3. Equation of water vapor transfer

$$\frac{\partial q}{\partial t} + \frac{u}{a \sin\theta}\frac{\partial q}{\partial \lambda} + \frac{v}{a}\frac{\partial q}{\partial \theta} + \bar{\omega}\frac{\partial q}{\partial \sigma} = \nu\nabla^2 q + \frac{g}{p_s}\frac{\partial \Pi_q}{\partial \sigma}; \qquad (1.4)$$

4. Equation of hydrostatic equilibrium, integrated along the vertical from $\sigma = 0$ to 1

$$H = H_s + \frac{R_B}{g}\int_{\sigma}^{1}\frac{T}{\xi}\,d\xi; \qquad (1.5)$$

5. Equation of continuity

$$\frac{\partial p_s}{\partial t} + \frac{1}{a \sin\theta}\left[\frac{\partial(p_s u)}{\partial \lambda} + \frac{\partial(p_s v \sin\theta)}{\partial \theta}\right] + \frac{\partial(p_s\bar{\omega})}{\partial \sigma} = 0. \qquad (1.5')$$

Integrating the left part of this equation over the entire mass of the atmosphere and using the boundary condition $\bar{\omega} = 0$ at $\sigma = 0$ and $\sigma = 1$, we obtain

$$\frac{\partial p_s}{\partial t} + \int_0^1 D\,d\xi = 0, \qquad (1.6)$$

where for brevity

$$D = \frac{1}{a \sin\theta}\left[\frac{\partial(p_s u)}{\partial \lambda} + \frac{\partial(p_s v \sin\theta)}{\partial \theta}\right]. \qquad (1.7)$$

Finally, integrating the equation of continuity over the vertical from 0 to σ and utilizing the known relationship

$$\omega = p_s\bar{\omega} + \sigma\frac{dp_s}{dt},$$

we obtain

$$\bar{\omega} = \frac{\sigma}{p_s} \int_0^1 D d\xi - \frac{1}{p_s} \int_0^\sigma D d\xi, \tag{1.8}$$

$$\omega = - \int_0^\sigma D d\xi + \sigma \left[\frac{u}{a \sin \theta} \frac{\partial p_s}{\partial \lambda} + \frac{v}{a} \frac{\partial p_s}{\partial \theta} \right]. \tag{1.9}$$

Thus, considering that the components of frictional stress and the fluxes of heat and moisture appearing on the right-hand sides of the original equations are functions of the dependent variables and also of a series of given "external" parameters of the problem, we obtain a closed system of eight equations, (1.1) to (1.9) for determining the eight unknown values u, v, T, q, p_s, H, $\bar{\omega}$, and ω. We note that the sink of water vapor resulting from condensation and precipitation, and the latent heat released in phase conversions of water are not calculated in the equations of water vapor transfer and heat flux. These processes are modeled in the scheme with the assistance of a special procedure, considered in paragraph 5.

2. BOUNDARY CONDITIONS

At the upper boundary of the atmosphere where $\sigma = 0$

$$\text{(a)} \quad \bar{\omega} = 0, \tag{1.10}$$

$$\text{(b)} \quad A = 0, \tag{1.11}$$

At the level of the underlying surface where $\sigma = 1$

$$\text{(a)} \quad u = v = \bar{\omega} = 0, \tag{1.12}$$

For the ocean, the temperature of its surface is assigned

$$T_s = T_s(\lambda, \theta), \tag{1.13}$$

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For dry land, the equation of heat balance for the underlying surface is used

$$\Pi_T + L\Pi_q + F = 0. \tag{1.14}$$

It is assumed that flows of mass, heat and moisture reduce to zero at the equator, that is, where $\theta = \frac{\pi}{2}$

$$v = 0, \quad \frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial q}{\partial \theta} = 0. \tag{1.15}$$

All functions are assumed periodic with λ.

3. INTEGRATING THE EQUATIONS OF THE MODEL

System (1.1) through (1.9) is solved by a method of separation. This method was first used by G. I. Marchuk for integrating equations of atmospheric dynamics, and it was he who demonstrated its effectiveness in a problem of short-term weather forecasting.[(4)] In agreement with the principle of this method, a system of equations of general atmospheric circulation at each step in time is solved in several stages. The following is the sequence of the calculations. In the first stage, the changes in the meteorological elements caused by radiational heat flux, by vertical turbulent exchange and by interaction of the atmosphere with the underlying surface are calculated. The values thus obtained are used as initial data for solving a system of advection equations describing the transfer of meteorological elements along a trajectory of the flow, with horizontal large-scale diffusion included. Then, by solving a problem of adjustment, dynamic agreement of the resultant fields is produced. Finally, in the last stage, the cloudiness is determined, and the resultant fields of temperature and moisture are corrected, with consideration of the released heat of condensation and of the processes of convection.

4. SELECTION OF A GRID

All variable values are determined at nodes of a latitudinal/longitudinal grid with an increment $\Delta\lambda$ along latitude and an increment $\Delta\theta$ along longitude.

In applying a given model, $\Delta\lambda = \Delta\theta = 5°$ is assumed, giving a total of 1297 nodes, including the pole, at a level in a hemisphere. Seven levels, corresponding to values of σ equal to 0.2, 0.3, 0.5, 0.7, 0.85, 0.99 and 1.00 are used. The coordinates of each node are determined by the index i, taken along a latitude to the east of the Greenwich meridian; by the index j, taken along a meridian in a southerly direction; and by the index k, taken along a vertical from the upper boundary of the atmosphere. Here, the step along the vertical is variable in size, depending on σ. Forecasting is accomplished with a time step Δt, with the index n determining the number of such steps. For the grid-point value of a function, the usual notation is introduced

$$f^{12}_{i,j,k} = f(i\Delta\lambda,\ j\Delta\theta,\ \sigma_k,\ n\Delta t),$$

where $i = 0,1,\ldots,I;\ j = 0,1,\ldots,J;\ k = 1,2,\ldots,K.$

The variable step along vertical $\Delta\sigma_k$ is equal to

$$\Delta\sigma_k = \sigma_k - \sigma_{k-1}. \tag{1.16}$$

In what follows, any of the indexes may be omitted when this does not cause incorrect interpretation of the equation.

II. FLUXES OF RADIANT ENERGY

It must be kept in mind, when developing a method for calculating radiational heat fluxes to be used in numerical models of the general atmospheric circulation and long-term weather forecasting, that hydrodynamic methods give limited information on the state of the atmosphere. This is due both to the usually applied low vertical resolution in the atmosphere, and to the fact that certain parameters on which the radiation flux depends cannot be obtained from a hydrodynamic method. Of particular relevance are the characteristics of scattering processes in the atmosphere and the albedos of the underlying surface and of the clouds. Cloudiness has a special significance in calculating radiation fluxes. The means for calculating cloudiness now used in many studies[5,6] demands knowledge of the degree of cloud cover. At the present time, this can be established only approximately, for example, by determining statistical relations between values of cloudiness obtained from observations and values of moisture or vertical velocity, which are calculated in a numerical method.[7,8]

Thus, it turns out that parameters known only approximately enter the calculation of radiation flux. In this connection, questions arise about the precision of calculation. The first: What sort of error is introduced by inadequate knowledge of the magnitude of the radiation fluxes? The second: If the requirements for precision in calculating the fluxes of radiation are known, what precision is required when assigning other basic parameters?

It follows that the precision necessary for calculating the fluxes of radiation depends on the sensitivity of the entire hydrodynamic model to variations in the flux of energy.

An answer to the second question may be given only after numerical experiments with a hydrodynamic model including radiation heat fluxes.

In view of certain individual experiments, the first of the questions posed above can be studied independently of the hydrodynamic model.

1. FLUXES OF SHORTWAVE RADIATION

The method for calculating radiation fluxes suggested by Manabe and Strickler is well known and is used. However, their plan has an essential inadequacy, since it disregards the transformation of the radiation field resulting from the absorption by aerosols and the effect of multiple scattering of shortwave radiation.

In an article[9] a proposal is made to use the work of K. S. Shifrin, I. N. Minin, O. A. Avaste and N. P. Piatovskaia[10-12] to construct a method for calculating the fluxes of shortwave radiation, taking into account the influence of multiple scattering.

We shall consider that part of the spectrum from 0.29 to 0.72μ. For a more complete consideration of factors acting in the atmosphere, we shall introduce absorption by ozone into the method for calculating the fluxes of total and direct radiation.

Equation (2.1) is used for calculating fluxes to total radiation at the level z in the interval 0.29 to 0.72μ:

$$Q_z = \int_{0.29}^{0.72} J_0(\lambda) B^*(\lambda) e^{-\sec i [k_\lambda \omega_z]} \, d\lambda. \tag{2.1}$$

Here $J_0 d\lambda$ is the intensity of solar radiation in the given part of the spectrum arriving at the upper boundary of the atmosphere; $B^*(\lambda)$ is the function converting the flux J_0 to radiance at the level z with repeated scattering[(2)*] taken into account:

$$B^*(\lambda) = \frac{2R_z(\lambda) \cos i}{4 + (3 - \kappa_1)(1 - r)\tau_z^\infty(\lambda)}, \tag{2.2}$$

where

$$R_z = 1 + \frac{3}{2} \cos i + \left(1 - \frac{3}{2} \cos i\right) e^{-\frac{a}{z} \sec i}. \tag{2.3}$$

*Editors' note: Apparently reference [12] was intended.

The parameter κ_1 characterizes the integration of the scattering function

$$\kappa_1 = \frac{3}{2}\int_0^{\pi} \kappa(\gamma)\sin\gamma\cos\gamma\, d\gamma,$$

where γ is the angle of scattering, κ is the scattering function, and r is the albedo of the underlying surface.

The optical density of the atmosphere is calculated according to the equation

$$\tau_z^{\infty} = \tau_{0,p}(\lambda)e^{-\alpha z} + \tau_{0,a}(\lambda)e^{-\beta z}, \tag{2.4}$$

where $\tau_{0,p}$ is the Rayleigh spectral optical density, $\tau_{0,a}$ is the aerosol spectral optical density, $\alpha = 0.125$ 1/km, $\beta = 0.898$ 1/km.[10]

In Eq. (2.1), k_λ is the logarithmic coefficient of ozone absorption according to Vigroux,[13] ω_z is the total content of ozone from the upper boundary of the atmosphere to a given level z, according to Elterman (model 1964),[14] and i is the zenith angle of the sun.

The flux of direct solar radiation, S, through the horizontal surface, taking into account single scattering and the absorption by ozone in the spectral interval 0.29 to 0.72μ, is calculated according to the equation

$$S_z = \int_{0.29}^{0.72} J_0(\lambda)\cos i\, e^{-\sec i[\tau_z^{\infty}(\lambda)+k_\lambda\omega_z]}\, d\lambda. \tag{2.5}$$

The fluxes of total and direct radiation at level z in the spectral interval 0.72 to 5.0μ are calculated by the equations

$$Q_z = \int_{0.72}^{5.0} J_0(\lambda)B^*(\lambda)T(X_{H_2O})T(X_{CO_2})\, d\lambda, \tag{2.6}$$

$$S_z = \int_{0.72}^{5.0} J_0(\lambda)\cos i\, e^{-\sec i[\tau_z^{\infty}(\lambda)]}\, T(X_{H_2O})T(X_{CO_2})\, d\lambda. \tag{2.7}$$

The transmission function T(X), entering expressions (2.6) and (2.7), is calculated from the approximation equation[10]

$$T(X) = 1 - \frac{X}{a(\lambda)X + b(\lambda)}. \tag{2.8}$$

Coefficients a and b for various parts of the spectrum for H_2O and CO_2 are cited in the work of K. S. Shifrin and O. A. Avaste.[10] The calculation of the masses of the absorbing atmospheric constituents is made according to the equation

$$X_k = \sqrt{W_k \sec i \overline{(p + l)^n} \Big|_{z_k}^{\infty}}, \tag{2.9}$$

where

$$\overline{(p + l)^n} \Big|_{z_k}^{\infty} = \frac{W_k'}{W_k}; \tag{2.10}$$

$$W_k = \int_{z_k}^{\infty} \rho_{\Pi} \, dz; \tag{2.11}$$

$$W_k' = \int_{z_k}^{\infty} \rho_{\Pi} (p + l)^n \, dz; \tag{2.12}$$

n is the coefficient expressing the influence of pressure variation on the value of the absorbing mass (n = 0.3 for H_2O and n = 0.4 for CO_2); ρ_{Π} is the density of the absorbing material; W_k^{∞} is the content of absorbing material in a column of air from the upper boundary of the atmosphere to the level z_k.

Thus, with the help of expressions (2.9) and (2.10), the suspended absorbing masses of water vapor are calculated.

The equations cited in Ref. 10 are used for calculating the absorbing masses of CO_2.

It is necessary to know the value of the flux of reflected radiation at various levels in the atmosphere.

The intensity of the reflected flux is calculated from the equation

$$B(\theta, \varphi, i) = \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} J_0(\lambda) r(\lambda) B^*(\lambda) T_1(\theta, \varphi, i, \lambda)\, d\lambda. \qquad (2.13)$$

For the flux of reflected radiation, we have

$$R(i) = \int_0^{2\pi} \int_0^{\pi/2} B(\theta, \varphi, i) \cos\theta \sin\theta \, d\theta \, d\varphi, \qquad (2.14)$$

where θ is the angle of reflection, φ is azimuth, and T_1 is a function describing the radiation absorption in the atmosphere.

Disregarding the dependence on azimuth, in the spectral interval 0.29 to 0.72μ, the function T_1 is given in the form

$$T_1 = e^{-[k_\lambda \omega_k^\infty \sec i + (\tau_l^k + k_\lambda \omega_l^k) \sec \theta]}. \qquad (2.15)$$

In the interval 0.72 to 5.0μ, the following expression was used for T_1:

$$T_1(\theta, i, \lambda) = e^{-\tau_l^k \sec \theta}\, T\left[X'_{H_2O}(\theta, i, \lambda)\right] T\left[X'_{CO_2}(\theta, i, \lambda)\right]. \qquad (2.16)$$

The single-stage scattering of the ray reflected at the angle θ and the absorption by water vapor and carbon dioxide of the incident and reflected ray is calculated in Eq. (2.15).

The absorbing masses X'_{H_2O} and X'_{CO_2} are calculated for the entire path traversed by the ray from the upper boundary of the atmosphere to level z_k in accordance with the equation

$$X' = X_l^{\infty}(i) + X_l^{k}(\theta), \quad (2.17)$$

where X_l^{∞} is the absorbing mass in the path of the ray from the upper boundary of the atmosphere to the reflecting surface l, X_l^k is the absorbing mass in the path of the reflected ray from the reflecting surface l to the level z_k.

The system of symbols used and the trajectory of the ray to the level z_k are presented in Fig. 1. The function $T(X')$ is calculated according to Eq. (2.8). The optical density τ_l^k is computed by the equation

$$\tau_l^k = \tau_l^{\infty} - \tau_k^{\infty}, \quad (2.18)$$

with the help of Eq. (2.4).

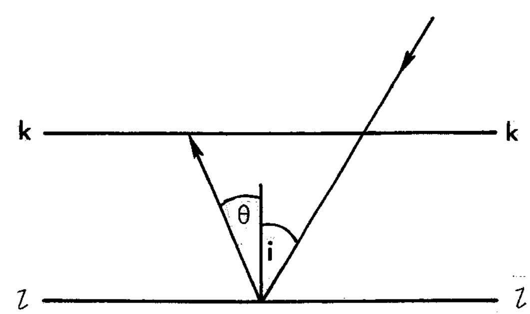

Fig. 1 — System of symbols

In (2.14) we perform a numerical integration over θ. Here, following the study,(7)* we separate the interval of integration into four parts: 0 to 30, 30 to 50, 50 to 70, 70 to 90°. Then, disregarding the dependence of opacity on azimuth, we obtain

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*Editors' note: Apparently reference [9] was intended.

$$R = 2\pi \left\{ B(20°, i) \int_0^{30°} \cos\theta \sin\theta \, d\theta + B(40°, i) \int_{30°}^{50°} \cos\theta \sin\theta \, d\theta \right.$$

$$\left. + B(60°, i) \int_{50°}^{70°} \cos\theta \sin\theta \, d\theta + B(80°, i) \int_{70°}^{90°} \cos\theta \sin\theta \, d\theta \right\}. \tag{2.19}$$

As a result, after calculating the integrals for the flux of reflected radiation, we obtain the following expression:

$$R = 2\pi \left\{ B(20°, i)0.0670 + B(40°, i)0.1115 \right.$$

$$\left. + B(60°, i)0.1500 + B(80°, i)0.1710 \right\}. \tag{2.20}$$

We shall assume the albedo of the reflecting surface to be independent of λ. In this case, the intensity of the ray reflected at angle θ is calculated by numerical integration over the spectrum.

For $\theta = 20°$

$$B(20°, i) = \frac{r}{\pi} \left\{ J_0(0.72)\Delta\lambda B^*(0.72)T_1(20°, i, 0.72) \right.$$

$$+ J_0(0.76)\Delta\lambda B^*(0.76)T_1(20°, i, 0.76)$$

$$\left. + \ldots + J_0(4.5)\Delta\lambda B^*(4.5)T_1(20°, i, 4.5) \right\}. \tag{2.21}$$

Expressions for $B(40°, \lambda)$; $B(60°, \lambda)$; $B(80°, \lambda)$ are written analogously.

Aerological observations made by co-workers in the Department of Atmospheric Physics at Leningrad State University were used to verify the accuracy of the method proposed for calculating fluxes of shortwave radiation according to the suggestion of K. Ya. Kondratyev. Data from four flights were used for initial data. Observations were made at 30 km above the earth's surface. Three flights (11 July 1964,

22 July 1964, and 21 October 1965) were made in cloudless conditions. During one flight (23 October 1964), the upper boundary of the clouds was at a height of 1.8 to 2 km. Values of the radiation fluxes, the height of the sun, and the meteorological characteristics were recorded during the ascent of the balloon. The albedo of the underlying surface, in the region of observation, was determined from ground observations.

Thus, the following measured values were used in the calculations:

1. Pressure values (mm) at 32 levels.
2. The values of water vapor pressure (mm) at the same levels.
3. A series of heights z (km), corresponding to the eight levels for which flux values were calculated.
4. A series of elevations of the sun.
5. Albedo of the underlying surface.

The calculations were made with the value of the solar constant 1.94 cal cm^{-2} min^{-1}.[15]

The values of the calculated and the actual fluxes of total and direct radiation on a horizontal surface, as well as the reflected radiation fluxes, are presented in Figs. 2 to 4.

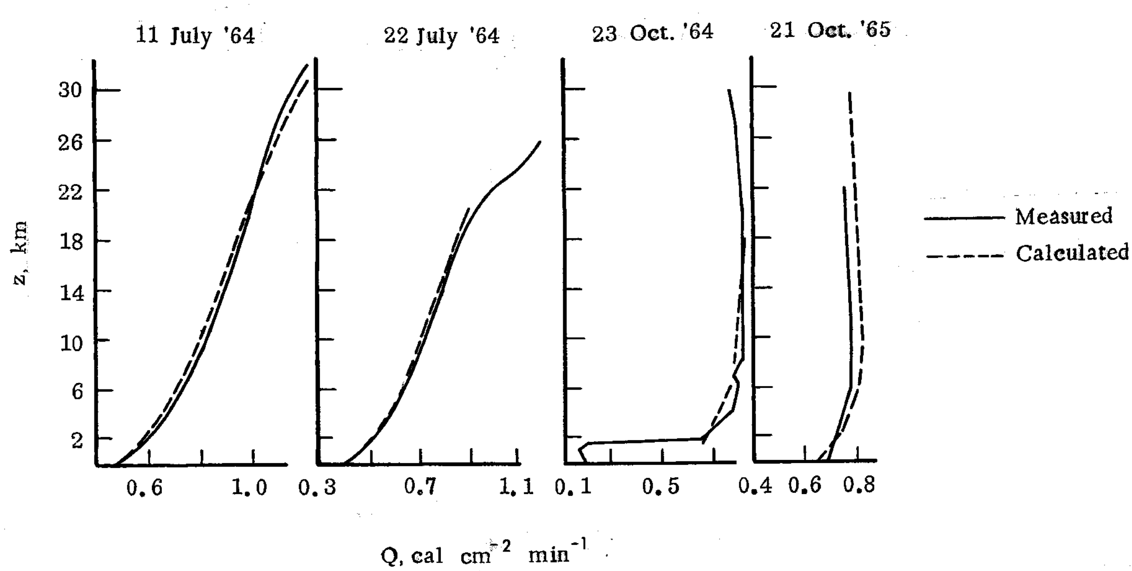

Fig.2 — Flows of total radiation on a horizontal surface

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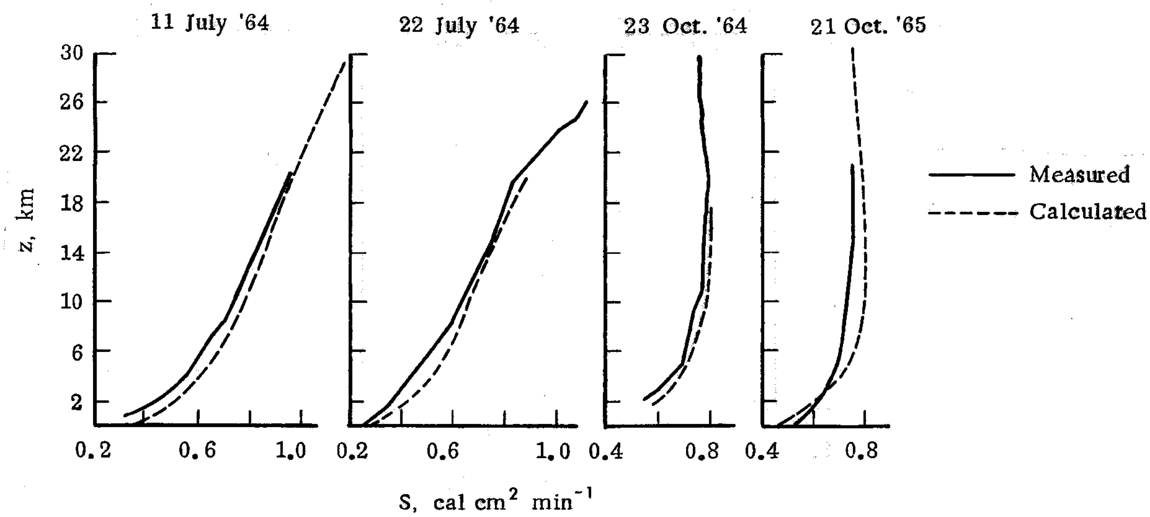

Fig.3 — Fluxes of direct radiation through a horizontal surface

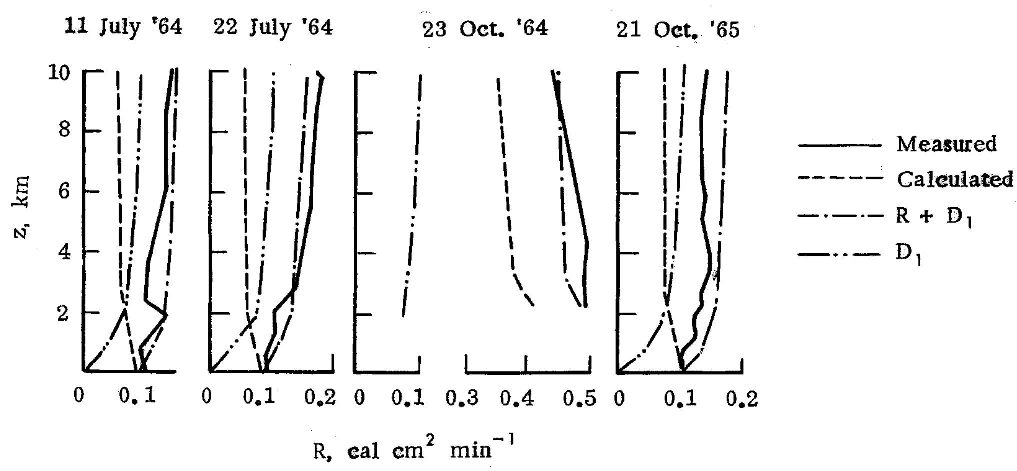

Fig.4 — Fluxes of reflected radiation

Comparison of the curves shows satisfactory agreement between calculated and measured values. The greatest discrepancies are noted in the reflected flux values. This may be explained as follows. The flux of reflected radiation, consisting of the flux reflected by the earth's surface and the flux scattered backward by a layer of haze in the atmosphere between the earth's surface and a given level, designated as D_1, was measured in the course of observation.

Equations (2.20) and (2.21) give only the first component of the flux. Its second component is found if we calculate the haze by the equations suggested by I. N. Minin and K. S. Shifrin. However, those calculations are quite lengthy. We therefore used calculations made by N. P. Piatovskaia and K. S. Shifrin[11] for different conditions of the underlying surface, and selected those which corresponded most closely to the albedo values of the underlying surface used in our calculations. Since opacity in haze depends significantly on albedo, approximation of this component may introduce errors in the value R.

The shortwave balance, $F_1 = R - Q$, in cloudless conditions cited in Table 1 was calculated from the values of total and reflected radiation. For comparison, the shortwave balance according to observational data is also presented.

It is likewise interesting to compare the values of the radiation flux convergence in the layers.

Profiles of the flux convergence, $-\Delta F_1/\Delta z$, are presented in Fig. 5, where ΔF_1 was calculated as the difference in values of shortwave balance at two levels.* The value of $-\Delta F_1/\Delta z$ is identified with the middle of the layer.

For quantitative evaluation of the discrepancies in the fluxes, the ratios $S_{calc.}/S_{meas.}$ and $Q_{calc.}/Q_{meas.}$ (Table 2) and the ratios $F_{1\ calc.}/F_{1\ meas.}$ and $\Delta F_{1\ calc.}/\Delta F_{1\ meas.}$ (Table 3) were determined.

The negative value of flux is without physical meaning. It was obtained as a result of the nonsynchronous observations at different levels, connected with a change in the height of the sun.

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*Editors' note: The values plotted in Fig. 5 are not in complete agreement with the data of Table 1.

Table 1

MEASURED AND CALCULATED VALUES OF SHORTWAVE RADIATION

z, km	22 July 1964		11 July 1967		23 Oct. 1964		21 Oct. 1965	
	F_1 Cal	F_1 Meas	F_1 Cal	F_1 Meas	F_1 Cal	F_1 Meas	F_1 Cal	F_1 Meas
0	-0.3155	-0.2830	-0.3920	-0.357	-0.1850	-0.1850	-0.5347	-0.5710
2.200	...	...	...	...	...	...	-0.5830	-0.5910
2.500	...	...	-0.4732	-0.4560	...	...	...	...
2.600	-0.3828	-0.4200	...	...	...	...	...	...
4.000	...	...	...	...	-0.2707	-0.2610	-0.6073	-0.6090
5.000	-0.4770	-0.4860	-0.5437	-0.5290	...	...	...	...
8.400	...	...	...	...	-0.3293	-0.3520	-0.6278	-0.6480
8.600	...	...	-0.6273	-0.6020	...	...	...	...
10.000	-0.5356	-0.5630	-0.6542	-0.6300	-0.3396	-0.3780	-0.6288	-0.6370

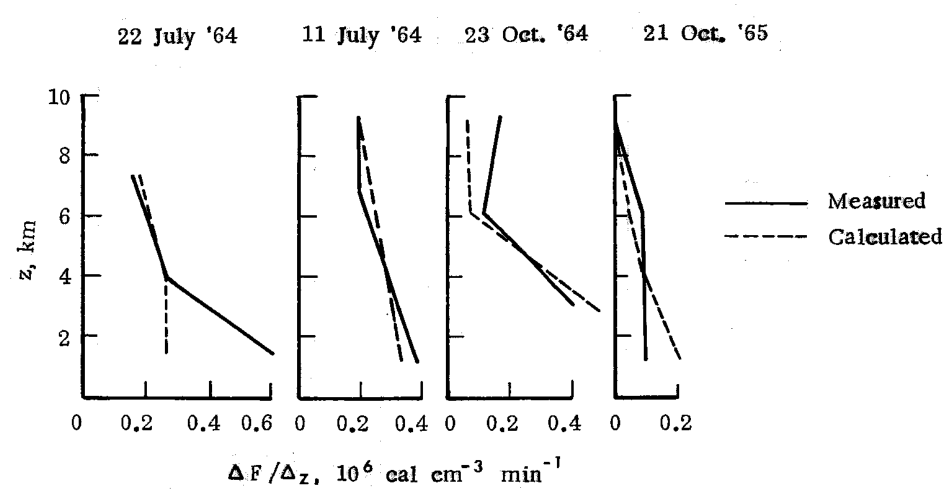

Fig.5 — Radiation flux convergence
in individual levels

Table 2

COMPARISON OF CALCULATED AND MEASURED VALUES OF FLUXES

21 Oct. 1965		
z, km	$\frac{S_{cal}}{S_{meas}}$	$\frac{Q_{cal}}{Q_{meas}}$
0	0.87	0.942
1.0	0.97	1.00
1.6	1.00	1.01
2.2	1.02	1.02
3.0	1.02	1.02
4.0	1.04	1.02
5.0	1.04	1.02
7.0	1.07	1.03
8.4	1.05	1.02
10.0	1.05	1.03
14.0	1.05	1.05
26.0	...	...
30.0	...	...

23 Oct. 1964		
z, km	$\frac{S_{cal}}{S_{meas}}$	$\frac{Q_{cal}}{Q_{meas}}$
0	...	...
1.0	...	...
1.6	...	...
2.2	1.09	0.99
2.6	1.08	0.98
3.0	1.07	0.97
3.3	1.04	0.95
4.0	1.04	0.97
5.0	1.00	0.96
7.0	1.01	0.98
8.4	1.01	0.97
10.0	1.00	0.97
16.0	0.99	0.98
20.0	0.99	1.00

22 July 1964		
z, km	$\frac{S_{cal}}{S_{meas}}$	$\frac{Q_{cal}}{Q_{meas}}$
0	1.00	1.05
1.3	1.14	0.95
2.0	1.15	0.98
2.6	1.16	0.98
3.6	1.16	0.98
5.0	1.12	0.97
7.0	1.08	0.97
9.0	1.06	0.97
10.0	1.05	0.97
16.0	1.00	0.97
20.0	1.02	0.97

11 July 1964		
z, km	$\frac{S_{cal}}{S_{meas}}$	$\frac{Q_{cal}}{Q_{meas}}$
0	1.24	1.08
1.3	1.28	1.06
2.0	1.19	1.05
2.5	1.14	1.03
3.5	1.11	1.01
4.0	1.07	0.98
5.0	1.10	1.00
6.0	1.06	0.98
8.0	1.05	0.99
8.6	1.06	1.01
10.0	1.05	1.00
11.0	1.03	0.98
15.0	1.01	0.98
27.0	...	1.01
30.0	...	1.03

Table 3

COMPARISON OF CALCULATED AND MEASURED VALUES OF FLUXES

22 July 1964		
z, km	$\frac{F_{cal}}{F_{meas}}$	$\frac{\Delta F_{cal}}{\Delta F_{meas}}$
0	1.1148	...
		0.419
2.6	0.9114	...
		0.974
5.0	0.9198	...
		1.143
10.0	0.9513	...
...	...	...

11 July 1964		
z, km	$\frac{F_{cal}}{F_{meas}}$	$\frac{\Delta F_{cal}}{\Delta F_{meas}}$
0	1.0980	...
		0.818
2.50	1.0377	...
		0.965
5.00	1.0278	...
		1.14
8.60	1.0420	...
		0.960
10.00	1.0384	...

23 Oct. 1964		
z, km	$\frac{F_{cal}}{F_{meas}}$	$\frac{\Delta F_{cal}}{\Delta F_{meas}}$
2.2	1.0	...
		1.1277
4.0	1.0371	...
		0.6482
8.4	0.9355	...
		0.3969
10.0	0.8984	...
...	...	...

21 Oct. 1965		
z, km	$\frac{F_{cal}}{F_{meas}}$	$\frac{\Delta F_{cal}}{\Delta F_{meas}}$
0	0.9364	...
		2.4147
2.2	0.9865	...
		1.350
4.0	0.997	...
		0.525
8.4	0.9688	...
		0.0909
10.0	0.9871	...

Table 2 shows that the discrepancy in values of fluxes is within the limits of 0 to 20 percent. At one point, the discrepancy amounts to 28 percent. The greatest differences are in the lower layers of the atmosphere up to 2 to 2.5 km. Apparently this circumstance may be partially explained by the disregard of the radiation absorption by aerosols in the calculation method.

Although the problem of aerosol absorption has not been adequately investigated up to the present, indirect indications of its influence do exist.[16] The problem has not received careful research.

From Table 3, it follows that discrepancies in the values of flux can be quite significant. Analysis has shown that large errors are found in cases of small values of radiation heat fluxes.

Thus, comparison shows that the present method for calculating the fluxes, S, Q, and R of shortwave radiation may be used for calculation on observational data in actual individual cases and in a model of the general atmospheric circulation.

The equations cited above were used to explain the influence of the underlying surface albedo, the scattering function, and the averaging of the zenith angle, and the effect of limited information about moisture on the value of the radiation flux in the spectral interval 0.7 to 5μ.

For variations in the albedo of the earth's surface, values of radiation flux with albedo values 0.16, 0.25, 0.6, and 0.8 are considered in a numerical experiment.

Fluxes of total and reflected radiation, effective flux F_1 at a given level z, the flux convergence in the layer ΔF_1, and the relationship of fluxes with different albedo values are presented in Table 4. From the table, it follows that the influence of the albedo of the underlying surface on the value of radiation flux convergence in the individual layers of the atmosphere is not great. Basically, the albedo influence is strongest in the lower atmospheric levels. Examples in the table relating to individual actual cases of water vapor distribution shows that a change of albedo from 0.16 to 0.8 leads to a difference of flux convergence in the lower atmospheric level of a total of 25 percent. Thus, it follows that demands for precise assignment

Table 4

COMPARISON OF FLUXES FOR DIFFERENT ALBEDOS

Upper Boundary of Sounding 1, km	Level, km	Layer km	Albedo 0.16 Q	R	F_1	ΔF_1	Albedo 0.25 Q	R	F_1	ΔF_1
10	0.0		0.2433	0.03918	-0.2041		0.2445	0.06112	-0.1834	
		0-2.5				-0.0795				-0.0818
	2.5		0.3164	0.03276	-0.2836		0.3166	0.04873	-0.2652	
		2.5-5.0				-0.0547				-0.0553
	5.0		0.3700	0.03164	-0.3383		0.7202	0.04657	-0.3205	
		5.0-8.6				-0.0605				-0.0606
	8.6		0.4302	0.03136	-0.3988		0.4303	0.04612	-0.3811	
		8.6-10.0				-0.0179				-0.179
	10.0		0.4481	0.03132	-0.4168		0.4481	0.04606	-0.3989	
30	0.0		0.2457	0.03931	-0.2064		0.2469	0.06172	-0.1852	
		0-8				-0.1746				-0.1778
	8.0		0.4126	0.03156	-0.3810		0.4126	0.04955	-0.3630	
		8-15				-0.0677				-0.0678
	15.0		0.4801	0.03140	-0.4487		0.4801	0.04929	-0.4308	
		15-27				-0.1084				-0.1084
	27.0		0.5884	0.03133	-0.5571		0.5884	0.04918	-0.5392	
		27-30				-0.0374				-0.0374
	30.0		0.6258	0.03132	-0.5945		0.6258	0.04918	-0.5766	

Upper Boundary of Sounding 1, km	Level, km	Layer km	Albedo 0.6 Q	R	F_1	ΔF_1	Albedo 0.8 Q	R	F_1	ΔF_1	$\Delta F_1(0.25)/\Delta F_1(0.16)$	$\Delta F_1(0.6)/\Delta F_1(0.16)$	$\Delta F_1(0.8)/\Delta F_1(0.16)$
10	0.0		0.2490	0.1494	-0.09960		0.2518	0.2014	0.0504				
		0-2.5				-0.0922				-0.0988	1.029	1.160	1.243
	2.5		0.3176	0.1258	-0.1958		0.3182	0.1695	-0.1487				
		2.5-5.0				-0.0578				-0.0584	1.011	1.057	1.068
	5.0		0.3706	0.1215	-0.2491		0.3708	0.1637	-0.2071				
		5.0-8.6				-0.0613				-0.0622	1.002	1.008	1.017
	8.6		0.4305	0.1204	-0.3101		0.4307	0.1621	-0.2686				
		8.6-10.0				-0.0179				-0.0178	0.989[a]	1.000	0.983
	10.0		0.4483	0.1202	-0.3281		0.4484	0.1621	-0.2863				
30	0.0		0.2515	0.1509	-0.1006		0.2542	0.2034	-0.0508				
		0-8				-0.1912				-0.1990	1.017	1.093	1.138
	8.0		0.4129	0.1212	-0.2918		0.4131	0.1633	-0.2498				
		8-15				-0.0679				-0.0681	1.001	1.001	1.006
	15.0		0.4802	0.1205	-0.3597		0.4803	0.1625	-0.3179				
		15-27				-0.1084				-0.1084	1.000	1.000	1.000
	27.0		0.5884	0.1203	-0.4681		0.5884	0.1621	-0.4263				
		27-30				-0.0375				-0.0375	1.000[a]	1.003	1.003
	30.0		0.6258	0.1202	-0.5056		0.6259	0.1621	-0.4638				

[a]Variations in values are connected with calculation errors.

of albedo need not be severe for the calculation of radiation flux convergence. For example, selection of an albedo equal to 0.25 instead of 0.16 is permissible, as is a replacement of the albedo values 0.6 and 0.8 by their average value 0.7.[29]

Another important aspect of the methodology of calculating fluxes is the necessity of evaluating errors, caused by the scarcity of original data on moisture that are used for calculating the absorbing masses. Of the 32 known values that describe the profile of moisture, six were selected. The remaining 26 values, which were used for calculating the fluxes, were obtained by interpolation.

Thus, radiation fluxes were calculated in two ways. In the first way--absorption by the water vapor mass was calculated according to the 32 values of moisture taken from the observational data. In the second way--the absorbing masses were found according to interpolated values.

For interpolation, an exponential dependence of moisture on σ was used, where $\sigma = p/p_s$ represents an approximate linear dependence of pressure on height.

The values obtained for the fluxes are presented in Table 5. Comparison of these values shows that the use of interpolated values of moisture and pressure leads to a discrepancy in the values of the fluxes of direct and total radiation not exceeding several percent. Table 5 contains results of calculations for two actual cases with a relatively smooth profile of moisture. Under actual conditions, singularities in the distribution of moisture usually occur with the appearance of clouds, in which case the absorbing water vapor masses must be calculated individually over and under the cloudy layers.

Calculations with values of the parameter κ_1 equal to 0.6, 1.2, and 2.0 were made to explain the influence of the form of the scattering function. Results show that the discrepancy in the value of total radiation amounts to 5 to 8 percent.

The conclusions obtained in this section make it possible to use approximate values of the albedo of the underlying surface in calculations of the flux of shortwave radiation, and to utilize a small number of levels in the vertical in calculating fluxes of radiation.

Table 5

OBSERVED AND INTERPOLATED VALUES OF RADIATION FLUXES

Level, km	Case 1				Case 2			
	S		Q		S		Q	
	With Interpolation	Without Interpolation	With Interpolation	Without Interpolation	With Interpolation	Without Interpolation	With Interpolation	Without Interpolation
0	0.2886	0.2864	0.3451	0.3426	0.1664	0.1653	0.2124	0.2110
1.6	0.3717	0.3702	0.3911	0.3896	0.2323	0.2316	0.2527	0.2520
2.6	0.3882	0.3959	0.4089	0.4067	0.2750	0.2705	0.2847	0.2800
3.5	0.4221	0.4107	0.4191	0.4177	0.2976	0.2932	0.3038	0.2993
5.0	0.4264	0.4258	0.4307	0.4301	0.3238	0.3223	0.3279	0.3264
7.0	0.4336	0.4343	0.4365	0.4372	0.3511	0.3491	0.3539	0.3519
16.0	0.4256	0.4270	0.4266	0.4279	0.4321	0.4334	0.4330	0.4344
20.0	0.4202	0.4197	0.4206	0.4202	0.4822	0.4809	0.4828	0.4815

2. FLUXES OF LONGWAVE RADIATION

Fluxes of longwave radiation are calculated according to well known equations.[6,17]

For fluxes at level z, directed downward, under cloudless conditions, we use the expression

$$A_z^\infty = E[T(m)] + \int_{E(m)}^{E(M)} D(u - m)dE[T(u)]. \qquad (2.22)$$

For fluxes directed upward

$$B_z^s = E[T(m)] - \int_0^{E(m)} D(m - u)dE[T(u)]$$

$$+ (1 - d) \int_0^{E(M)} D(u + m)dE[T(u)], \qquad (2.23)$$

where

$$m = \int_0^z \rho_\pi dz, \tag{2.24}$$

$$M = \int_0^\infty \rho_\pi dz, \tag{2.25}$$

and where D is the transmission function, E is the radiation of a perfectly black body, m is the mass of the absorbing medium from the earth's surface to level z, M is the amount of the absorbing material in the entire mass of the atmosphere, and ρ_π is the density of the absorbing medium.

The integral function, proposed by F. N. Shekhter, which considers the dependence of the absorption on the amount of water vapor, carbon dioxide, and ozone, is used for the transmission function.

A particular variant of Shekhter's transmission function[18] has the form

$$D(H_2O,\ CO_2,\ O_3) = D(H_2O,\ CO_2) - A(O_3), \tag{2.26}$$

where $D(H_2O, CO_2)$ is the transmission function for water vapor and carbon dioxide together, and $A(O_3)$ is the absorption function for ozone.

The expression for $D(H_2O, CO_2)$ is constructed so that the mass of water vapor is the only parameter, with the influence of CO_2 considered automatically on the basis of the functional connection between the masses of H_2O and CO_2:

$$D(H_2O,\ CO_2) = D(W) = D^*(W) - \Delta D, \tag{2.27}$$

where

$$D^* = 0.471e^{-0.695\sqrt{W}} + 0.529e^{-8.94\sqrt{W}}. \tag{2.28}$$

Here W is the effective absorbing mass of water vapor, determined with the help of the expression

$$W = \frac{1}{\sqrt{p_0}} \int_0^z \rho_\pi \sqrt{p}\, dz. \tag{2.29}$$

The correct values ΔD are the following:

W	0.0001	0.0003	0.001	0.003	0.01	0.03	0.1	0.3	1	3
100ΔD . . .	7.4	8.2	8.5	8.9	9.2	7.4	5.2	2.2	0.2	0

The calculation of the effective absorbing masses is performed according to the equations obtained by R. L. Kagan[19] from (2.29), using the equation of state and the hydrostatic equation adapted for calculation according to data on the water vapor and pressure at various levels:

$$W_i = \sum_{j=1}^{i-1} \Delta W_j, \tag{2.30}$$

$$\Delta W_j = 0.01 \left(\frac{l_j}{\sqrt{p_j}} + \frac{l_{j+1}}{\sqrt{p_{j+1}}} \right) (p_j - p_{j+1}), \tag{2.31}$$

with $i = 2,3,4,\ldots$; $j = 1,2,3\ldots$ Here i is the number of levels considered in the calculation of absorbing masses.

The absorption function for ozone is represented by the following data:

m_{O_3} . . .	0.01	0.033	0.06	0.1	0.2	0.4	0.6
$100A(O_3)$.	0.4	0.9	1.6	2.3	3.3	4.1	4.4

Evaluation of the integrals in (2.22) and (2.23) according to Simpson's equation makes it possible to obtain approximate expressions for the fluxes.(6) The effective fluxes of longwave radiation F_2 at each level are calculated in the form $F_2 = B - A$.

3. CALCULATION OF FLUXES UNDER CLOUDY CONDITIONS

The method used for the consideration of cloudiness(5,6) consists of the following: the flux at level z is calculated as the sum of the fluxes relating to the cloudy and cloudless parts of the sky and is taken with corresponding weights.

In three-layered cloudiness, fluxes from all cloudy layers enter the flux at level z, in correspondence with the scheme presented in Fig. 6 as an example.

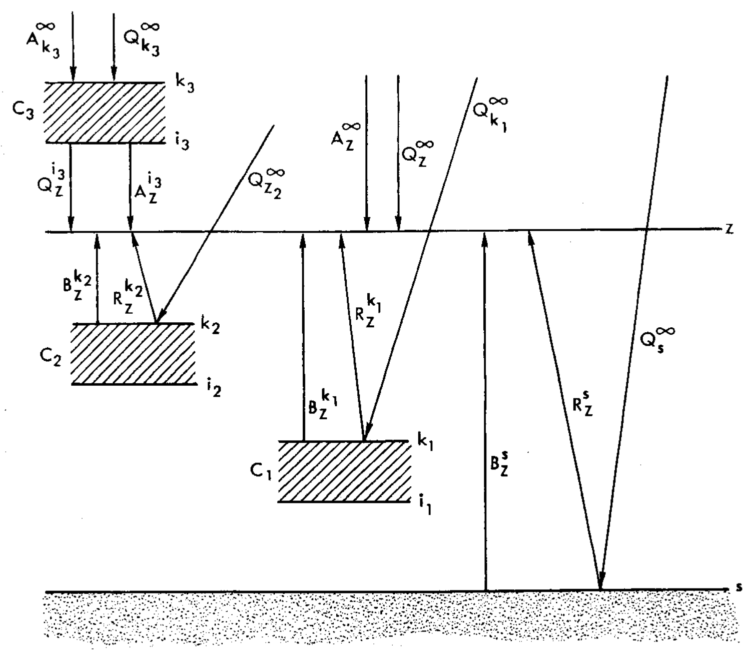

Fig.6 — Scheme of directed radiation fluxes in cloudy conditions

Generally, the flux from above, at level z, is calculated as the sum:

$$F_z^{\downarrow} = (A_z^{\infty} + Q_z^{\infty}) \left(1 - \sum_{i=1}^{3} \hat{C}_i\right) + \left(A_z^{i_1} + Q_z^{i_1}\right) \hat{C}_1$$

$$+ \left(A_z^{i_2} + Q_z^{i_2}\right) \hat{C}_2 + \left(A_z^{i_3} + Q_z^{i_3}\right) \hat{C}_3. \qquad (2.32)$$

Fluxes, directed upward from below are calculated according to the equation

$$F_z^{\uparrow} = (B_z^s + R_z^s)\left(1 - \sum_{i=1}^{3} \tilde{C}_i\right) + \left(B_z^{k_1} + R_z^{k_1}\right)\tilde{C}_1$$

$$+ \left(B_z^{k_2} + R_z^{k_2}\right)\tilde{C}_2 + \left(B_z^{k_3} + R_z^{k_3}\right)\tilde{C}_3. \qquad (2.33)$$

The meaning of the quantities entering expressions (2.32) and (2.33) is presented below:

$A_z^{i_1}$, $A_z^{i_2}$, $A_z^{i_3}$ are fluxes of longwave radiation from the lower boundary of clouds in the lower, middle, and upper strata respectively;

$B_z^{k_1}$, $B_z^{k_2}$, $B_z^{k_3}$ are fluxes of longwave radiation reaching level z from the upper boundary of clouds in the lower, middle, and upper strata respectively;

$Q_z^{i_1}$, $Q_z^{i_2}$, $Q_z^{i_3}$ are fluxes of shortwave radiation transmitted by clouds in the lower, middle, and upper strata;

$R_z^{k_1}$, $R_z^{k_2}$, $R_z^{k_3}$ are fluxes of shortwave radiation, reflected from the upper boundary of clouds in the lower, middle, and upper strata respectively and reaching level z;

$\hat{C}_i$ is the amount of cloudiness visible from below from given level z;

$\tilde{C}_i$ is the amount of cloudiness visible from above from given level z.

The following relations are used for recalculating actual cloudiness C_i visible from below or above, with i = 1 for clouds at the lower stratum, i = 2 for clouds of the middle stratum, and with i = 3 for clouds of the upper stratum.

$$C_1 = \hat{C}_1,$$

$$C_2 = \frac{\hat{C}_2}{1 - \hat{C}_1},$$

$$C_3 = \frac{\hat{C}_3}{1 - (\hat{C}_2 + \hat{C}_3)}; \tag{2.34}$$

$$C_1 = \frac{\tilde{C}_1}{1 - (\tilde{C}_2 + \tilde{C}_3)},$$

$$C_2 = \frac{\tilde{C}_2}{1 - \tilde{C}_3},$$

$$C_3 = \tilde{C}_3. \tag{2.35}$$

If one of the three cloud layers is absent, the corresponding value C_i is assumed equal to zero and equations (2.32) and (2.35) reduce to the appropriate particular case.

We calculate the effective flux in cloudy conditions as

$$F_z = F_z^{\uparrow} - F_z^{\downarrow}.$$

4. CALCULATION OF RADIATIONAL HEAT FLUX

To obtain an equation for calculating the fluxes with data at a limited number of layers, it is assumed that the effective flux F_1 may be presented as a quadratic function of the vertical coordinate in the form:

$$F_k = a + b(\sigma - \sigma_k) + c(\sigma - \sigma_k)^2.$$

We shall find expressions for the coefficients a, b, c in terms of the values of the fluxes at the levels k - 1, k + 1. If we introduce the designations

$$\sigma_{k+1} - \sigma_k = \Delta\sigma_{k+1} = p,$$

$$\sigma_k - \sigma_{k-1} = \Delta\sigma_k = q,$$

we obtain

$$b = \frac{q^2 F_{k+1} - p^2 F_{k-1} - F_k (q^2 - p^2)}{pq(p + q)},$$

since

$$\left(\frac{\partial F}{\partial \sigma}\right)_{\sigma=\sigma_k} = b,$$

and therefore

$$\left.\frac{\partial F}{\partial \sigma}\right|_{\sigma=\sigma_k} = \frac{q}{p(p + q)} F_{k+1} - \frac{p}{q(p + q)} F_{k-1} - \frac{q - p}{pq} F_k.$$

The net radiational heating (the radiational heat flux convergence) Φ is expressed by the derivative $\partial F/\partial\sigma$ in the following manner: $\Phi = g/c_p P_s \ \partial F/\partial\sigma$. Fluxes are calculated from the following original data:

1. Distribution of moisture,
2. Distribution of pressure,
3. Albedo of the underlying surface and of clouds,
4. Absorbing properties of clouds.

III. INTEGRATION OF EQUATIONS OF THE MODEL

1. CONSIDERATION OF THE VALUES OF FRICTION, HEAT, AND MOISTURE FLUXES

Changes in the wind velocity, temperature, and moisture caused by vertical turbulent diffusion and the radiant flux of heat are described by the following equations

$$\frac{\partial u}{\partial t} = -\frac{g}{p_s}\frac{\partial \tau_\lambda}{\partial \sigma}, \tag{3.1}$$

$$\frac{\partial v}{\partial t} = -\frac{g}{p_s}\frac{\partial \tau_\theta}{\partial \sigma}, \tag{3.2}$$

$$\frac{\partial T}{\partial t} = \frac{g}{c_p p_s}\frac{\partial}{\partial \sigma}(\Pi_T + F), \tag{3.3}$$

$$\frac{\partial q}{\partial t} = \frac{g}{p_s}\frac{\partial \Pi_q}{\partial \sigma}. \tag{3.4}$$

We shall assume that the height of the planetary boundary layer coincides with the level σ_{k-2}, while within the boundary layer, two calculating levels are located* on σ_{k-1} and σ_k; outside of this layer the intensity of friction and turbulent heat fluxes are equal to zero. Consequently,

$$\frac{\partial u_k}{\partial t} = \frac{\partial v_k}{\partial t} = 0; \quad \frac{\partial q_k}{\partial t} = 0, \tag{3.5}$$

$$\frac{\partial T_k}{\partial t} = \frac{g}{p_s c_p}\left(\frac{\partial F}{\partial \sigma}\right)_{\sigma_k} = \Phi(\sigma_k). \tag{3.6}$$

---

*Level σ_k coincides with the surface of the earth.

As a vertical coordinate, we shall take the geometric height z in place of σ, and we shall write the basic equations for the boundary layer in the form

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial \tau_\lambda}{\partial z}, \tag{3.1'}$$

$$\frac{\partial \bar{u}}{\partial t} = - u_*^2 \cos \alpha, \tag{3.1''}$$

$$\frac{\partial \bar{v}}{\partial t} = - u_*^2 \sin \alpha, \tag{3.2''}$$

$$\frac{\partial \bar{T}}{\partial t} = - k u_* T_* - \frac{1}{c_p \rho} [F(h) - F(\delta)], \tag{3.3''}$$

$$\frac{\partial \bar{q}}{\partial t} = - k u_* q_*. \tag{3.4''}$$

Here α is the angle between the direction of the vector of the frictional force near the underlying surface and the tangent to the circle of latitude. We also introduce the definition

$$\bar{\chi}(u, v, T, q) \equiv \int_h^\delta \chi dz.$$

Approximating χ as a polynomial of the fourth degree in z, satisfying all the boundary conditions (3.8), we obtain

$$\bar{\chi} = a_1 \chi(H) + a_2 \chi(h) + a_3 \chi(\delta), \tag{3.12}$$

where

$$a_1 = (\delta - h) c \left(\frac{\delta - h}{\delta - H} \right)^3,$$

$$a_2 = (\delta - h) \left(\frac{1}{4} - c \right),$$

$$a_3 = (\delta - h)\left\{\frac{3}{4} + c\left[1 - \left(\frac{\delta - h}{\delta - H}\right)^3\right]\right\},$$

$$c = \frac{\delta - h}{20(H - h)}.$$

If $(H - h) \ll \delta$, then with adequate precision we have

$$a_1 = (\delta - h)(c - 0.15),$$

$$a_2 = (\delta - h)(0.25 - c),$$

$$a_3 = 0.15(\delta - h).$$

Now, considering the boundary conditions (3.9) and the expressions (3.11) and (3.12), it is possible to write a system of equations (3.1″) to (3.4″) individually for dry land and ocean in the following form:

For dry land:

$$\frac{\partial}{\partial t}\left[a_1 u(H) + a_2 u(h)\right] = -u_*^2 \cos\alpha, \tag{3.13}$$

$$\frac{\partial}{\partial t}\left[a_1 v(H) + a_2 v(h)\right] = -u_*^2 \sin\alpha, \tag{3.14}$$

$$\frac{\partial}{\partial t}\left[a_1 T(H) + a_2 T(h)\right] = -\frac{F(\delta)}{c_p\rho} - a_3\Phi(\delta). \tag{3.15}$$

For the ocean:

$$\frac{\partial}{\partial t}\left[a_1 u(H) + a_2 u(h)\right] = -u_*^2 \cos\alpha, \tag{3.16}$$

$$\frac{\partial}{\partial t}\left[a_1 v(H) + a_2 v(h)\right] = -u_*^2 \sin\alpha, \tag{3.17}$$

$$\frac{\partial}{\partial t}\left[a_1 T(H) + a_2 T(h)\right] = -ku_*T_* - \frac{1}{c_p\rho}\left[F(h) - F(\delta)\right] - a_3\Phi(\delta), \tag{3.18}$$

$$\frac{\partial}{\partial t}\,[a_1 q(H) + a_2 q(h)] = -\,k u_* q_*. \tag{3.19}$$

Since within the surface layer, the direction of the wind velocity changes little and coincides with the direction of the vector of frictional force, it is possible to assume that

$$\cos\alpha = u(H)/V(H),$$

$$\sin\alpha = v(H)/V(H), \tag{3.20}$$

where $V^2 = u^2 + v^2$.

The system of equations (3.13) to (3.15) contains seven unknowns and the system (3.16) to (3.19) eleven unknowns. For completing these systems we shall use the known expressions for the profiles of wind velocity, temperature and moisture:[21]

$$u(h) = \frac{u_*}{k}\,[f(h/L) - f(z_0/L]\cos\alpha,$$

$$T(h) - T(z_0) = T_*[f(h/L) - f(z_0/L)],$$

$$q(h) - q(z_0) = q_*[(h/L) - f(z_0/L)], \tag{3.21}$$

(here L is the length scale

$$L = \frac{u_*^2}{b^2(\beta T_* + 0.61 g q_*)},$$

and $\beta = g/T$ is the buoyancy parameter) and four empirical relationships, obtained in Ref. 22.

- for ocean and dry land:

$$-b u_* T_* = \frac{10^{-2} l V^2(\delta)}{\beta}\,\mu N(\mu), \tag{3.22}$$

- for dry land

$$u_* = V(\delta)M(\mu)\left[0.089 - 0.00356 \ln \frac{V(\delta)}{l z_0}\right], \tag{3.23}$$

- for ocean

$$u_* = 0.030V(\delta)M(\mu), \tag{3.24}$$

$$-bu_*q_* = 10^{-2}V(\delta)[q(z_0) - q(\delta)]N(\mu). \tag{3.25}$$

Here

$$N(\mu) = [0.274 + 0.0167\mu^{1/3}]^2, \tag{3.26}$$

$$M(\mu) = 1 + 0.8(\frac{\mu}{1000}) - 0.3(\frac{\mu}{1000})^2 + 0.08(\frac{\mu}{1000})^3, \tag{3.27}$$

$$\mu = \beta \frac{T(z_0) - T(\delta) - 8.4}{l V(\delta)}. \tag{3.28}$$

The empirical relationships (3.22) to (3.25) are not valid for lower latitudes. The question of the structure of the boundary layer near the equator still requires both theoretical and experimental studies.

2. INTEGRATION OF THE ADVECTION EQUATIONS

We shall write the system of advection equations in the following form:

$$\frac{\partial u}{\partial t} + \frac{u}{a \sin\theta}\frac{\partial u}{\partial \lambda} + \frac{v}{a}\frac{\partial u}{\partial \theta} + \bar{\omega}\frac{\partial u}{\partial \sigma} = \nu\left[\frac{1}{a^2 \sin^2\theta}\frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{a^2}\frac{\partial^2 u}{\partial \theta^2}\right], \tag{3.29}$$

$$\frac{\partial v}{\partial t} + \frac{u}{a \sin\theta}\frac{\partial v}{\partial \lambda} + \frac{v}{a}\frac{\partial v}{\partial \theta} + \bar{\omega}\frac{\partial v}{\partial \sigma} = \nu\left[\frac{1}{a^2 \sin^2\theta}\frac{\partial^2 v}{\partial \lambda^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \theta^2}\right], \tag{3.30}$$

$$\frac{\partial T}{\partial t} + \frac{u}{a \sin \theta} \frac{\partial T}{\partial \lambda} + \frac{v}{a} \frac{\partial T}{\partial \theta} \bar{\omega} \frac{\partial T}{\partial \sigma} = \nu \left[\frac{1}{a^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial^2 T}{\partial \theta^2} \right] \tag{3.31}$$

$$\frac{\partial q}{\partial t} + \frac{u}{a \sin \theta} \frac{\partial q}{\partial \lambda} + \frac{v}{a} \frac{\partial q}{\partial \theta} \bar{\omega} \frac{\partial q}{\partial \sigma} = \nu \left[\frac{1}{a^2 \sin^2 \theta} \frac{\partial^2 q}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial^2 q}{\partial \theta^2} \right] \tag{3.32}$$

On the right-hand sides of these equations small terms of the form $\frac{\operatorname{ctg} \theta}{a^2} \frac{\partial u}{\partial \theta}$ are omitted, since even for polar regions these components are one or two orders less than the remaining terms.

In turn, this system is divided into three sequentially solved problems. First, values of u, v, T, and q are advected along the circle of latitude, then along the meridian, and finally along the vertical. For brevity of notation we shall introduce the so-called standard system of coordinates according to the relationships

$$\begin{aligned} dx &= a \sin \theta \, d\lambda, \\ dy &= a d\theta, \end{aligned} \tag{3.33}$$

and the vector function

$$\underset{\sim}{\Phi} = \begin{Vmatrix} u \\ v \\ T \\ q \end{Vmatrix}. \tag{3.34}$$

Then the system (3.26) to (3.29) assumes the form

$$\frac{\partial \underset{\sim}{\Phi}}{\partial t} + u \frac{\partial \underset{\sim}{\Phi}}{\partial x} + v \frac{\partial \underset{\sim}{\Phi}}{\partial y} = \nu \left[\frac{\partial^2 \underset{\sim}{\Phi}}{\partial x^2} + \frac{\partial^2 \underset{\sim}{\Phi}}{\partial y^2} \right]. \tag{3.35}$$

This vector equation is also solved by a method of splitting. It is divided into two equations in such a manner that derivatives are present only along one spatial variable. First we solve the equation

$$\frac{\partial \underset{\sim}{\Phi}}{\partial t} + u \frac{\partial \underset{\sim}{\Phi}}{\partial x} = \nu \frac{\partial^2 \underset{\sim}{\Phi}}{\partial x^2}, \tag{3.36}$$

during the time interval $t_n \le t \le t_n + \Delta t$, then, taking the function Φ obtained as a result of the solution of (3.36) for the initial distribution, we solve the equation

$$\frac{\partial \underset{\sim}{\Phi}}{\partial t} + v \frac{\partial \underset{\sim}{\Phi}}{\partial y} = \nu \frac{\partial^2 \underset{\sim}{\Phi}}{\partial y^2}, \tag{3.37}$$

during the same interval in time $t_n + \Delta t$. Since the algorithm for solving these equations is the same, we shall limit ourselves to a consideration of (3.36).

In schemes for numerically modeling the general circulation of the atmosphere and ocean given by Leith[23] and Crowley,[24] explicit finite-difference schemes of second-order accuracy along spatial coordinates for one scalar quasi-linear equation of the form (3.35) were proposed, and it was shown that these schemes are conditionally stable. However, with integration for a long period of time, they give significant errors in phase. Fromm[25] and Crowley[26] developed more complex difference schemes, making it possible to lessen the errors in amplitude and phase of the numerical solution, as a result of increasing the order of approximation of the spatial derivatives.

Unlike the referenced studies, we shall consider not one, but a system of equations (3.36) and find their common solution. In Ref. 27, a precise solution to the system (3.36) is found in integral form; by using it, it is possible to construct diverse finite-difference schemes.

We introduce a new scalar function ϕ and vector-function $\underset{\sim}{\Psi}$ in agreement with the nonlinear relationships

$$u = -2\nu \frac{\partial \ln \phi}{\partial x}; \qquad \underset{\sim}{\Psi} = \underset{\sim}{\Phi} \cdot \phi. \tag{3.38}$$

It is not difficult to show,[27] that ϕ and $\underset{\sim}{\Psi}$ satisfy the equation of heat conduction, for example $\frac{\partial \underset{\sim}{\Psi}}{\partial t} - \nu \frac{\partial^2 \underset{\sim}{\Psi}}{\partial x^2} = 0$. Considering the solution

of this equation on an unbounded straight line during the interval of time $t_n \leq t \leq t_n + \Delta t$ with initial conditions for ϕ

$$\phi^n = \exp\left[-\frac{1}{2\nu}\int_0^x u^n d\xi\right], \tag{3.39}$$

and for $\underset{\sim}{\Psi}$

$$\underset{\sim}{\Psi}^n = \underset{\sim}{\Phi}^n \cdot \phi^n, \tag{3.40}$$

following from (3.38), we obtain for $x = 0$

$$\phi^{n+1} = \frac{1}{2\sqrt{\pi\nu\Delta t}}\int_{-\infty}^{\infty} \phi^n \exp\left(-\frac{\xi^2}{4\nu\Delta t}\right) d\xi. \tag{3.41}$$

$$\underset{\sim}{\Psi}^{n+1} = \frac{1}{2\sqrt{\pi\nu\Delta t}}\int_{-\infty}^{\infty} \underset{\sim}{\Psi}^n \exp\left(-\frac{\xi^2}{4\nu\Delta t}\right) d\xi. \tag{3.42}$$

Hence, on the basis of (3.38), we obtain

$$\underset{\sim}{\Phi}_0^{n+1} - \underset{\sim}{\Phi}_0^n = \frac{\displaystyle\int_{-\infty}^{\infty} [\underset{\sim}{\Phi}^n - \underset{\sim}{\Phi}_0^n]\ \phi^n \exp\left(-\frac{\xi^2}{4\nu\Delta t}\right) d\xi}{\displaystyle\int_{-\infty}^{\infty} \phi^n \exp\left(-\frac{\xi^2}{4\nu\Delta t}\right) d\xi}. \tag{3.43}$$

Here, $\underset{\sim}{\Phi}_0^n$ is the value of the function ϕ at the origin.

From (3.43) it is possible to obtain a solution at the nodes of the grid considered. We shall place the source at node i of the grid and assume there is a linear dependence of u on x in calculating the function ϕ according to (3.39)

$$u = c + bx. \tag{3.44}$$

We introduce the designation

$$J_k = \int_{-\infty}^{\infty} \xi^k \phi^n \exp\left(-\frac{\xi^2}{4\nu\Delta t}\right) d\xi. \tag{3.45}$$

Evaluating this integral by parts, we obtain

$$J_k = 2\nu\Delta t(k-1)J_{k-2} - 2\nu\Delta t \int_{-\infty}^{\infty} \xi^{k-1} \frac{\partial\phi^n}{\partial\xi} \exp\left(-\frac{\xi^2}{4\nu\Delta t}\right) d\xi. \tag{3.46}$$

On the basis of (3.38), substituting $\frac{\partial\phi^n}{\partial\xi}$ for $-\frac{1}{2\nu}\phi^n u^n$ and using (3.44) and (3.45), we come to the following relationship:

$$(1 + b\Delta t)J_k = -c\Delta t J_{k-1} + 2\nu\Delta t(k-1)J_{k-2}. \tag{3.47}$$

Hence

$$\begin{aligned}
\bar{J}_1 &= -\beta, \\
\bar{J}_2 &= \beta^2 + \gamma, \\
\bar{J}_3 &= -\beta(\beta^2 + 3\gamma), \\
\bar{J}_4 &= \beta^4 + 6\beta^2\gamma + 3\gamma^2.
\end{aligned} \tag{3.48}$$

Here $\quad \bar{J}_k = \frac{J_k}{J_0}, \quad \beta = \frac{c\Delta t}{1 + b\Delta t}, \quad \gamma = \frac{2\nu\Delta t}{1 + b\Delta t}.$

Now, if we introduce the function $\underset{\sim}{\Phi}^n$ in the form of a portion of the Taylor series in the vicinity of x = 0;

$$\underset{\sim}{\Phi}^n - \underset{\sim}{\Phi}^n_i = \sum_{k=1}^{N} \left(\frac{\partial^k \underset{\sim}{\Phi}^n}{\partial x^k}\right)_i \frac{x^k}{k!},$$

and then substitute this expression in (3.43) and use (3.45), we obtain

$$\underset{\sim}{\Phi}_i^{n+1} - \underset{\sim}{\Phi}_i^n = \sum_{k=1}^{N} \frac{\bar{J}_k}{k!}\left(\frac{\partial^k \underset{\sim}{\Phi}^n}{\partial x^k}\right)_i . \tag{3.49}$$

From Eq. (3.49) a series of known approximations may be obtained as a particular case. We shall consider several such methods.

1. $c = u_i$, $b = 0$, $N = 1$.

   In this case, in agreement with (3.49)

$$\underset{\sim}{\Phi}_i^{n+1} - \underset{\sim}{\Phi}_i^n = - u_i^n \Delta t \left(\frac{\partial \underset{\sim}{\Phi}^n}{\partial x}\right)_i , \tag{3.50}$$

   or

$$\underset{\sim}{\Phi}_i^{n+1} - \underset{\sim}{\Phi}_i^n = \begin{cases} \alpha(\underset{\sim}{\Phi}_i^n - \underset{\sim}{\Phi}_{i-1}^n) & \text{with } u_i^n \geq 0 \\ \alpha(\underset{\sim}{\Phi}_{i+1}^n - \underset{\sim}{\Phi}_i^n) & \text{with } u_i^n < 0 \end{cases} \tag{3.51}$$

$$\alpha = \frac{u_i^n \Delta t}{\Delta x}. \tag{3.52}$$

   What is obtained is an alternate scheme of first-order approximation.

2. $c = u_i^n$, $b = 0$, $N = 2$.

   Then

$$\underset{\sim}{\Phi}_i^{n+1} - \underset{\sim}{\Phi}_i^n = - u_i^n \Delta t \left(\frac{\partial \underset{\sim}{\Phi}}{\partial x}\right)_i^n + \left[\frac{(u_i^n \Delta t)^2}{2} + \nu\Delta t\right]\left(\frac{\partial^2 \underset{\sim}{\Phi}}{\partial x^2}\right)_i^n , \tag{3.53}$$

   or, substituting derivatives of $\underset{\sim}{\Phi}^n$ according to the equation of central differences and introducing the designation

$$r = \frac{\nu\Delta t}{(\Delta x)^2}, \tag{3.54}$$

we obtain

$$\underset{\sim}{\Phi}_i^{n+1} - \underset{\sim}{\Phi}_i^n = -\frac{\alpha}{2}(\underset{\sim}{\Phi}_{i+1}^n - \underset{\sim}{\Phi}_{i-1}^n)$$

$$+\left(\frac{\alpha^2}{2} + r\right)(\underset{\sim}{\Phi}_{i+1}^n - 2\underset{\sim}{\Phi}_i^n + \underset{\sim}{\Phi}_{i-1}^n). \tag{3.55}$$

This equation was obtained differently in the work of Leith[23] for the case $\nu = 0$.

3. $c = u_i^n$, $b = 0$, $N = 4$.

In this case, approximating derivatives with fourth-order precision,[28] we obtain

$$\underset{\sim}{\Phi}_i^{n+1} - \underset{\sim}{\Phi}_i^n = -\frac{\alpha}{12}[8(\underset{\sim}{\Phi}_{i+1}^n - \underset{\sim}{\Phi}_{i-1}^n) - (\underset{\sim}{\Phi}_{i+2}^n - \underset{\sim}{\Phi}_{i-2}^n)]$$

$$-\frac{1}{24}(\alpha^2 + 2r)\ [30\underset{\sim}{\Phi}_i^n - 16(\underset{\sim}{\Phi}_{i+1}^n + \underset{\sim}{\Phi}_{i-1}^n) + (\underset{\sim}{\Phi}_{i+2}^n + \underset{\sim}{\Phi}_{i-2}^n)]$$

$$-\frac{1}{12}(\alpha^3 + 6\alpha r)\ [-2(\underset{\sim}{\Phi}_{i+1}^n - \underset{\sim}{\Phi}_{i-1}^n) + \underset{\sim}{\Phi}_{i+2}^n - \underset{\sim}{\Phi}_{i-2}^n]$$

$$+\frac{1}{24}(\alpha^4 + 12\alpha^2 r + 12r^3)\ [6\underset{\sim}{\Phi}_i^n - 4(\underset{\sim}{\Phi}_{i+1}^n + \underset{\sim}{\Phi}_{i-1}^n)$$

$$+(\underset{\sim}{\Phi}_{i+2}^n + \underset{\sim}{\Phi}_{i-2}^n)]. \tag{3.56}$$

This scheme is presented in the work of Crowley[26] for the case $\nu = 0$.

Now we shall consider the case $b = (\frac{\partial u}{\partial x})_i^n$, $c = u_i^n$. Assuming $N = 2$ in Eq. (3.49), we obtain

$$\underset{\sim}{\Phi}_i^{n+1} - \underset{\sim}{\Phi}_i^n = -\frac{\alpha'}{2}(\underset{\sim}{\Phi}_{i+1}^n - \underset{\sim}{\Phi}_{i-1}^n)$$

$$+\left(\frac{\alpha'^2}{2} + r\right)(\underset{\sim}{\Phi}_{i+1}^n - 2\underset{\sim}{\Phi}_i^n + \underset{\sim}{\Phi}_{i-1}^n). \tag{3.57}$$

$$\alpha' = \frac{u_i^n \Delta t}{\Delta x \left(1 + \frac{u_{i+1} - u_{i-1}}{2\Delta x} \Delta t\right)}.$$

It is not difficult to show that this scheme satisfies the requirement of preserving the first moment of the equations of motion.

We note that with the aid of (3.49), schemes can be obtained possessing the property of preserving also the second moments for the equations of motion. Thus considering

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$

and assuming in (3.49) $c = \frac{1}{3} (u_{i+1} + u_i + u_{i-1})$ and $b = 0$, we obtain the Arakawa scheme,(33)* satisfying the conditions of preserving the average values of u and u^2 on a given segment with appropriate boundary conditions.

We shall now consider the equation for vertical advection

$$\frac{\partial \underset{\sim}{\Phi}}{\partial t} = - \bar{\omega} \frac{\partial \underset{\sim}{\Phi}}{\partial \sigma}. \tag{3.58}$$

Taking $\bar{\omega}$ at the moment $t = n\Delta t$, it is possible to obtain the equation, analogous to (3.55)

$$\underset{\sim}{\Phi}_k^{n+1} = \underset{\sim}{\Phi}_k^n + \frac{\bar{\omega}_k^n \Delta t (\bar{\omega}_k^n \Delta t - \Delta\sigma_k)}{\Delta\sigma_{k+1} (\Delta\sigma_{k+1} + \Delta\sigma_k)} (\underset{\sim}{\Phi}_{k+1}^n - \underset{\sim}{\Phi}_k^n)$$

$$- \frac{\bar{\omega}_k^n \Delta t (\bar{\omega}_k^n \Delta t + \Delta\sigma_{k+1})}{\Delta\sigma_k (\Delta\sigma_{k+1} + \Delta\sigma_k)} (\underset{\sim}{\Phi}_k^n - \underset{\sim}{\Phi}_{k-1}^n),$$

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*Editors' note: See also A. Arakawa, "A Computational Design for Long-Term Numerical Integration of the Equations of Fluid Motion: Two-Dimensional Incompressible Flow, Part I," *J. Comput. Phys.*, 1(1): 119-143, 1966.

where

$$\Delta\sigma_k = \sigma_k - \sigma_{k-1},$$

$$\Delta\sigma_{k+1} = \sigma_{k+1} - \sigma_k.$$

For a uniform grid ($\Delta\sigma_k = \Delta\sigma_{k+1} = \Delta\sigma_{k-1}$), we obtain the formula (3.55) in which $\nu = 0$ and

$$\alpha = \frac{\bar{\omega}_k^n \, \Delta t}{\Delta\sigma}.$$

3. SOLUTION OF A SYSTEM OF ADAPTATION EQUATIONS

At this stage, the problem of the dynamic adjustment of meteorological fields is resolved. The following system of equations is considered

$$\frac{\partial u}{\partial t} + l v = - \frac{g}{a \sin\theta} \frac{\partial H}{\partial \lambda} - \frac{R_B T}{p_s a \sin\theta} \frac{\partial p_s}{\partial \lambda}, \tag{3.59}$$

$$\frac{\partial v}{\partial t} - l u = - \frac{g}{a} \frac{\partial H}{\partial \theta} - \frac{R_B T}{p_s a} \frac{\partial p_s}{\partial \theta}, \tag{3.60}$$

$$H = H_s + \frac{R_B}{g} \int_\sigma^1 \frac{T}{\xi} \, d\xi. \tag{3.61}$$

$$\frac{\partial p_s}{\partial t} + \int_0^1 D d\xi = 0, \tag{3.62}$$

$$\frac{\partial T}{\partial t} - \frac{\kappa - 1}{\kappa} \frac{T\omega}{p_s \sigma} = 0. \tag{3.63}$$

Here and subsequently

$$l = \left(2\Omega + \frac{u}{a \sin\theta}\right) \cos\theta. \tag{3.64}$$

The relationship (1.9) must also be added to the system for the determination of the value of ω.

We reconstruct the first two equations of the system. For brevity, writing

$$f_u = \frac{g}{a \sin \theta} \frac{\partial H}{\partial \lambda} + \frac{R_B T}{p_s a \sin \theta} \frac{\partial p_s}{\partial \lambda}, \tag{3.65}$$

$$f_v = \frac{g}{a} \frac{\partial H}{\partial \theta} + \frac{R_B T}{p_s a} \frac{\partial p_s}{\partial \theta}, \tag{3.66}$$

we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} + l v &= - f_u, \\ \frac{\partial v}{\partial t} - l u &= - f_v. \end{aligned} \tag{3.67}$$

If we introduce the complex velocity $w = u + iv$ and the complex variable $F = f_u + if_v$, the system (3.67) may be written in the form of one equation

$$\frac{\partial w}{\partial t} - i l w = - F. \tag{3.68}$$

The solution of this equation has the form

$$w^{n+1} = w^n e^{i l \Delta t} - \int_{t_n}^{t_{n+1}} F e^{i l (t_{n+1} - \xi)} d\xi. \tag{3.69}$$

Removing the average value of F from under the integral over the segment (t_n, t_{n+1}), which we designate by F^*, we obtain

$$w^{n+1} = w^n e^{i l \Delta t} - \frac{iF^*}{l} (1 - e^{i l \Delta t}). \tag{3.70}$$

Separating the real and imaginary parts of the expression (3.70) we come to the following equalities:

$$u^{n+1} = u^n \cos \varepsilon - v^n \sin \varepsilon - \frac{1}{l} [f_u^* \sin \varepsilon - f_v^* (1 - \cos \varepsilon)],$$

$$v^{n+1} = u^n \sin \varepsilon + v^n \cos \varepsilon - \frac{1}{l} [f_u^* (1 - \cos \varepsilon) + f_v^* \sin \varepsilon]. \qquad (3.71)$$

where

$$\varepsilon = l\Delta t. \qquad (3.72)$$

The values of p_s and T entering f^* and f_v^* are calculated at the time $t_n + \Delta t/2$ by the obvious scheme. Thus, we determine

$$p_s^* = p_s^n - \frac{\Delta t}{2} \int_0^1 D^n d\xi, \qquad (3.73)$$

and

$$T^* = T^n + \frac{\Delta t}{2} \frac{\kappa - 1}{\kappa} \frac{T}{p_s} \frac{\omega^n}{\sigma}. \qquad (3.74)$$

According to the determined values of T^*, we determine

$$H^* = H_s + \frac{R_B}{g} \int_\sigma^1 \frac{T^*}{\xi} d\xi. \qquad (3.75)$$

Then p_s^*, H^* and T^* are substituted in (3.65) and (3.66) and with the determined values of f_u^* and f_v^* and the velocity field at time t_n, the velocity components u^{n+1} and v^{n+1} are found from (3.71). Then $p_s^{n+1} \sin \theta$ is determined according to the formula

$$p_s^{n+1} = p_s^n - \frac{\Delta t}{a \sin \theta} \int_0^1 \left[\frac{\partial}{\partial \lambda} (u^* p_s^*) + \frac{\partial}{\partial \theta} (v^* p_s^* \sin \theta) \right] d\xi. \qquad (3.76)$$

where
$$u^* = \frac{1}{2}(u^{n+1} + u^n),$$
$$v^* = \frac{1}{2}(v^{n+1} + v^n).$$

From the values u^{n+1}, v^{n+1} and p_s^{n+1} we obtain ω^{n+1} from (1.9). In the same manner, the determination of the velocity field and surface pressure at the particular time is established. It remains to determine dynamically the adjusted temperature field at moment t_{n+1}. This is calculated by the following equation

$$T^{n+1} = T^n + \frac{(\kappa - 1)\Delta t T^n}{2\kappa\sigma p_s}(\omega^n + \omega^{n+1}). \tag{3.77}$$

4. FINITE-DIFFERENCE APPROXIMATION OF DERIVATIVES

In calculating derivatives along spatial coordinates, a constant step is introduced along longitude

$$\Delta y = a\Delta\theta \approx 555 \text{ km} \tag{3.78}$$

and a variable step along latitude

$$\Delta x_j = x_j a \sin\theta_j \Delta\lambda. \tag{3.79}$$

Here x_j is an integral function which assumes the following values:

$$x_1 = 12,\ x_2 = 6,\ x_3 = 4,\ x_4 = 3,\ x_5 = \ldots = x_8 = 2,$$
$$x_9 = \ldots = x_{18} = 1. \tag{3.80}$$

Thus, the minimum value of the step Δx_j is equal to 390 km, and the maximum is 708 km. Derivatives along the horizontal are calculated with the aid of the central differences

$$\left(\frac{\partial f}{a\partial\theta}\right)_{i,j,k} \approx \frac{f_{i,j+1,k} - f_{i,j-1,k}}{2\Delta y},$$
$$\left(\frac{\partial f}{a \sin\theta\, \partial\lambda}\right)_{i,j,k} \approx \frac{f_{i+xj,j,k} - f_{i-xj,j,k}}{2\Delta x_j}. \tag{3.81}$$

The value of D is approximated in the following manner:

$$D_{i,j,k} = \frac{1}{2}\left\{\frac{(p_s u)_{i+xj,j,k} - (p_s u)_{i-xj,j,k}}{\Delta x_j} + \frac{(p_s v)_{i,j+1,k}\sin\theta_{j+1} - (p_s v)_{i,j-1,k}\sin\theta_{j-1}}{\Delta y \sin\theta_j}\right\}. \qquad (3.82)$$

Integrals appearing in the calculating formulas are computed by means of trapezoids.

We now proceed to the determination of the desired functions at the pole, which is a singular point of the equations being considered. Assuming that the vertical component of vorticity and the velocity divergence are bounded at the pole[30] we obtain with $\theta = 0$

$$u = \hat{u}(\sigma, t)\cos\lambda - \hat{v}(\sigma, t)\sin\lambda,$$

$$v = \hat{u}(\sigma, t)\sin\lambda + \hat{v}(\sigma, t)\cos\lambda. \qquad (3.83)$$

Here $\hat{u}$ and $\hat{v}$ are determined according to the values of the functions u and v on the circle of latitude closest to the pole, where these functions are calculated in the process of forecasting. For convenience of the calculations, the latitude $\Phi = 0$ is treated as the boundary of the area, containing 72 nodes. Components of the velocity vector are determined at these nodes by Eqs. (3.83), thanks to which the average value of the mass flow within the area at this boundary is equal to zero. Any scalar value is determined at the pole also by its value at the neighboring circle of latitude, and has one and the same value at all nodes of the boundary $\Phi = 0$.

IV. CALCULATING CONVECTION AND HEAT OF CONDENSATION. ESTABLISHING THE PARAMETERS OF CLOUDINESS

Considered in this paragraph is a method for establishing the amount of cloudiness (in tenths) and its boundaries, as well as a method for correcting the temperature, T, and the humidity, q, resulting from the process of condensation and convection.

1. FORECASTING CLOUDINESS

Cloudiness in three strata is forecast. Here it is assumed that cloudiness in the upper stratum is in the layer 500 to 300 mb, cloudiness in the middle stratum is in the layer 850 to 500 mb, and cloudiness in the lower stratum is in the layer 990 to 850 mb. We will denote the tenths of cloudiness in the upper stratum C_b, in the middle C_c, and in the lower C_h. The tenths of cloudiness in each stratum are determined by values of the average relative humidity and $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_3$ in the respective layer on the basis of the empirical graph of Smagorinsky,(31) establishing the following relations:

$$C_B = \begin{cases} 0 & \text{with} \quad \hat{f}_1 \le 0.60 \\ 3.25\,\hat{f}_1 - 1.95 & \text{with} \quad 0.60 < \hat{f}_1 \le 0.90 \\ 1 & \text{with} \quad 0.90 < \hat{f}_1 \end{cases}$$

$$C_C = \begin{cases} 0 & \text{with} \quad \hat{f}_2 \le 0.35 \\ 2\,\hat{f}_2 - 0.7 & \text{with} \quad 0.35 < \hat{f}_2 \le 0.85 \\ 1 & \text{with} \quad 0.85 < \hat{f}_2 \end{cases}$$

$$C_H = \begin{cases} 0 & \text{with} \quad \hat{f}_3 \le 0.25 \\ 1.72\,\hat{f}_3 - 0.43 & \text{with} \quad 0.25 < \hat{f}_3 \le 0.83 \\ 1 & \text{with} \quad 0.83 < \hat{f}_3 \end{cases} \qquad (4.1)$$

We shall term the values $\hat{f}_1 = 0.90$, $\hat{f}_2 = 0.85$, $\hat{f}_3 = 0.83$ critical ones and designate them f_{1cr}, f_{2cr}, f_{3cr}. Average values of the relative humidity in the layer are determined through the prognostic values of relative humidity according to the formulas:

$$\hat{f}_1 = \frac{1}{2}(f_2 + f_3),$$

$$\hat{f}_2 = \frac{1}{2}\left(\frac{3}{7} f_3 + f_4 + \frac{4}{7} f_5\right),$$

$$\hat{f}_3 = \frac{1}{2}(f_5 + f_6). \tag{4.2}$$

Here f_k are known functions of q_k, T_k:

$$f_k = \frac{q_k}{q_k^{max}(T_k)},$$

where $q_k^{max}(T_k)$ is determined by the Magnus* formula

$$q_k^{max}(T_k)\ \frac{3.8}{p_s\sigma_k} \exp\left\{1.71\ \frac{T_k - 273}{T_k - 38}\right\}.$$

Processing to a determination of the boundary of cloudiness, we note that the vertical resolution assumed in the given model is insufficient to obtain actual values of cloud density with adequate approximation. Therefore, intermediate levels are introduced

$$\sigma_{k+1/2} = \frac{1}{2}(\sigma_k + \sigma_{k+1}) \qquad k = 2,3,\ldots,K-2,$$

for which T, q and f are found by linear interpolation.

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*Editors' note: The intended expression is

$$q_k^{max}(T_k) = \frac{3.8}{p_s\sigma_k} \exp\left\{17.27\ \frac{T_k - 273}{T_k - 35.86}\right\}.$$

The following scheme for establishing the boundaries of the cloudiness is adopted. If it turns out that there is cloudiness in any stratum as a result of calculations from Eqs. (4.1) and (4.2), then:

a. In the event of unbroken cloudiness ($C = 1$), cloudiness is found on all successively situated layers (primary and intermediary) corresponding to the given stratum, for which $f > f_{cr}$ beginning at the level with maximum f;
b. In the event of broken cloudiness ($C < 1$), cloudiness is ascribed only to that level where f is a maximum.

From each level defined in such a manner, cloudiness spreads upward and downward to 1/4 the thickness of the primary layer, that is, the layer included between the two neighboring basic σ levels. Possible variations in the disposition of cloudiness are presented in Fig. 7. Primary levels between which cloudiness is found are represented by heavy lines. Thin lines indicate boundaries of cloudiness. An area occupied by cloudiness is cross-hatched, while cloudiness of various strata within the limits of one primary layer is indicated by cross-hatching with a different slant.

In this scheme, all levels were considered downward from above (from σ_2 to σ_{k-1}). If cloudiness of a particular stratum is noted at a given level, that level is no longer considered in the distribution of cloudiness in the lower-lying stratum, which excludes the possibility of the presence of cloudiness at two strata at the same time.

The scheme constructed for prognosis of cloudiness is completed by the following conditions:

a. In case $q_k^{max} < \dfrac{0.12}{p_3\sigma_k}$, cloudiness at level k is absent;

b. At level $K - 1$ cloudiness is always absent.

2. CONSIDERATION OF CONDENSATION AND CONVECTION

In the present scheme, processes of condensation and convection are considered individually in two stages. In the first stage, a

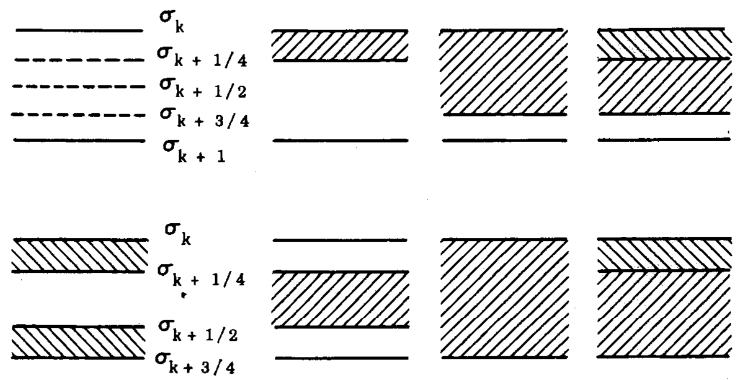

Fig.7 — Possible distribution of cloudiness along the vertical in a primary layer

correction made of the temperature and humidity fields as caused only by condensation in cloud layers; in the second stage, a correction is made as caused by convection, where it is assumed that the specific humidity in the cloud during the entire process of convection remains saturated.

<u>Consideration of condensation</u>. Condensation in layers of cloudiness lying in primary σ levels is considered. If the value of the relative humidity at level σ_k, containing cloudiness of the μ stratum, exceeds $f_{\mu cr}$, the heat of condensation is released, as a result of which the temperature T_k and the humidity q_k of the air change. Their new values T_k^* and q_k^* are calculated from the formulas:

$$q_k^* = f_{\mu cr}\left\{q_k^{max}(T_k) + \beta_k(T_k^* - T_k)\right\}, \tag{4.3}$$

$$T_k^* = T_k - \lambda C_\mu(q_k^* - q_k), \tag{4.4}$$

where

$$\beta_k = \frac{dq_k^{max}(T_k)}{dT}, \quad \lambda = \frac{L}{c_p}.$$

Equation (4.3) is obtained from the condensation condition

$$q_k^* = f_{\mu cr} q_k^{max}(T_k). \tag{4.5}$$

If $q_k^{max}(T_k)$ are separated into a series by stages $T_k^* - T_k$ and are limited by two primary members of the separation, which in the given case insures sufficient precision, then we will obtain Eq. (4.3).

Equation (4.4) is obtained in accordance with the suggestion of Smagorinsky that the process of condensation occurs so rapidly that only a local change in temperature takes place. Then the energy equation may be written in the form

$$c_p \frac{\partial T}{\partial t} \Delta t = - LC_\mu(q^* - q). \tag{4.6}$$

Substituting the increment of temperature with $T_k^* - T_k$, we obtain Eq. (4.4).

Unlike existing schemes for calculating the latent heat of condensation, here it is foreseen that condensation may occur even with broken cloudiness.

Calculation of convection. For calculating convection, a correction to the temperature and humidity are made if the vertical gradient of temperature in saturated air exceeds the moist adiabatic γ^{ma}, or in unsaturated air the dry adiabatic γ^{ad}. In each primary layer, the values γ^{ma} and γ^{ad}, calculated in the σ system, are assumed constant along the vertical within the limits of this layer. Temperature values entering the expressions for γ^{ma} and γ^{ad}, are selected as averages, according to the corresponding primary layer.

In order that the influence of any convective process may be expressed every time by a change in temperature and moisture in primary levels (since only the values T and q may be considered in other parts of the scheme) it is assumed that the temperature changes in proportion to γ^{ma} in the cloud and in proportion to γ^{ad} outside the cloud in each primary layer. In this case, if convection occurs, it completely envelops the primary layer or several such layers. We designate the amount of cloudiness occupying the upper, middle or lower part of the parimary layer by C_1, C_2 and C_3, respectively. Then, the condition for the absence of convection may be written as:

$$\frac{T_{k+1} - T_k}{\Delta\sigma_k} \le \gamma_k^{ad}(1 - C) + \gamma_k^{ma}C, \tag{4.7}$$

where

$$C = \frac{1}{4}(C_1 + 2C_2 + C_3).$$

Convection occurs in the event of non-fulfillment of the condition (4.7), as a result of which values of the temperature, T, and moisture, q, in the primary layers change. If convection envelops m primary layers, then new values of temperature T^* at corresponding primary levels may be defined from the system of equations:

$$\frac{T^*_{i+1} - T^*_i}{\Delta\sigma_i} = \gamma_i^{ad}(1 - C_i) + \gamma_i^{ma}C_i, \qquad i = k,\ldots, k + m-1, \tag{4.8}$$

$$\sum_{i=k}^{m-1} (T^*_i + T^*_{i+1})\Delta\sigma_i = \sum_{i=k}^{m-1} (T_i + T_{i+1})\Delta\sigma_i. \tag{4.9}$$

Equation (4.9) is obtained on the assumption that convection occurs without condensation. In this case[32] the condition is fulfilled:

$$\int_{\sigma_k}^{\sigma_n} (T - T^*)d\sigma = 0. \tag{4.10}$$

Evaluating this integral approximately by the trapezoidal equation, we obtain Eq. (4.9).

REFERENCES

1. Gandin, L. S., A. S. Dubov, and M. E. Shvets (eds.), *Theory of Climate. A Collection of Translated Articles*, Leningrad, Gidrometeoizdat, 1967.

2. Monin, A. S. (ed.), *The Dynamics of Large Scale Atmospheric Processes*, Proceedings of an international symposium, Moscow, 23-30 June 1965, Moscow, Nauka, 1967.

3. Gandin, L. S., and A. S. Dubov, *Numerical Methods of Short-Term Weather Forecasting*, Leningrad, Gidrometeoizdat, 1968.

4. Marchuk, G. I., *Numerical Methods in Weather Forecasting*, Leningrad, Gidrometeoizdat, 1967.

5. Manabe, S., and R. F. Strickler, "Thermal Equilibrium of an Atmosphere with a Convective Adjustment," *J. Atmos. Sci.*, Vol. 21, No. 4, 1964 [pp. 361-385].*

6. Dmitrieva-Arrago, L. R., "Calculation of Fluxes of Long-Wave Radiation under Cloudy Conditions," *Trudy, Main Geophysical Observatory*, Vol. 197, 1968.

7. Lewis, W., "Forecasting 700 mb Dewpoint Depression by a 3-Dimensional Trajectory Technique," *Monthly Weather Review*, Vol. 85, No. 9, 1957 [pp. 297-301].

8. Smagorinsky, J., "On the Dynamical Prediction of Large-Scale Condensation by Numerical Methods," [Geophysical] Monograph No. 5, *Physics of Precipitation*, American Geophysical Union, 1960.

9. Dmitrieva-Arrago, L. R., and L. V. Samoilova, "On Calculating the Vertical Distribution of Water Vapor, Albedo of the Underlying Surface and Changes in the Height of the Sun for Computations of Short-Wave Radiation in a Scheme of General Atmospheric Circulation," *Izvestiia, Academy of Science of the USSR, Physics of the Atmosphere and Ocean*, Vol. 6, No. 1, 1970.

10. Shifrin, K. S., and O. A. Avaste, "Fluxes of Short-Wave Radiation in Cloudless Atmosphere," Collection, *Studies on Physics of the Atmosphere*, No. 2, Tartu, 1960.

11. Shifrin, K. S., and N. P. Piatovskaia, "Field of Short-Wave Radiation over a Typical Underlying Surface," *Trudy, Main Geophysical Observatory*, Vol. 166, 1964.

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*Material in square brackets did not appear in original Russian text.

12. Shifrin, K. S., V. Iu. Kolomiitsev, and N. P. Piatovskaia, "Determining the Flow of Outgoing Short-Wave Radiation with the Aid of Artificial Earth Satellites," *Trudy, Main Geophysical Observatory*, Vol. 166, 1964.

13. Vigroux, E., "Contributions to Experimental Study of Ozone Absorption," *Ann. Phys.* [Paris], No. 8, 1954 [pp. 709-62].

14. Elterman, L., "Atmospheric Attenuation Model 1964," [Air Force Cambridge Research Laboratories, AFCRL 64-740], *Environmental Research Paper No. 46, 1964.*

15. Kondrat'ev, K. Ia., and G. A. Nikol'skii, "Variations in Solar Constant According to Data of Aerological Research 1962-1968," *Izvestiia, Academy of Science USSR, Physics of the Atmosphere and Ocean*, Vol. 6, No. 4, 1970.

16. Kastrov, V. G., "Measurement of Solar Radiation Absorption in the Free Atmosphere up to 3--5 km," *Trudy, Central Aerological Observatory*, Vol. 8, 1952.

17. Gandin, L. S., (and others), *Fundamentals of Dynamic Meteorology*, Leningrad, Gidrometeoizdat, 1955.

18. Shekhter, F. N., "About Radiation Diagrams," *Trudy, Main Geophysical Observatory*, Vol. 174, 1965.

19. Kagan, L. O., "About Calculating Flows of Thermal Radiation in a Cloudless Atmosphere," *Trudy, Main Geophysical Observatory*, Vol. 174, 1965.

20. Budyko, M. I. (ed.), *Heat Balance Atlas* [*of the Earth*], Main Geophysical Observatory, [Leningrad], 1955.

21. Monin, A. S., and A. M. Iaglom, *Statistical Hydromechanics*, Pt. 1, Chap. 4, Moscow, Nauka, 1965.

22. Dubov, A. S., and L. R. Orlenko, "About Calculating Processes in a Boundary Layer with Numerical Modeling of General Atmospheric Circulation," *Trudy, Main Geophysical Observatory*, Vol. 256, 1970.

23. Leith, C. E., "Numerical Simulation of the Earth's Atmosphere," *Methods in Computational Physics*, Vol. 4, [pp. 1-28, N. Y. Academic Press, 1965], (Russian translation in the collection, *Numerical Methods of Solving Problems of Dynamics of the Atmosphere and Ocean*, L. R. Dmitrieva-Arrago, L. V. Rukhovets, and M. E. Shneerov (eds.), Leningrad, Gidrometeoizdat, 1968.)

24. Crowley, W. P., "A Global Numerical Ocean Model: Part 1," *J. Comput. Phys.*, Vol. 3, No. 1, 1968 [pp. 111-147].

25. Fromm, J. E., "A Method for Reducing Dispersion in Convective Difference Schemes," *J. Comput. Phys.*, Vol. 3, No. 2, 1968.

26. Crowley, W. P., "Numerical Advection Experiments," *Monthly Weather Review*, Vol. 96, No. 1, 1968 [pp. 1-11].

27. Shvets, M. E., and V. P. Meleshko, "About a Numerical Algorithm for Solving Systems of Equations of Atmospheric Hydrodynamics," *Izvestiia, Academy of Sciences, USSR, Physics of the Atmosphere and Ocean*, Vol. 1, No. 91, 1965.

28. Kollatts, L., *Numerical Methods of Solving Differential Equations*, Moscow, Inostrannia Literatura, 1953.

29. Dmitrieva-Arrago, L. R. (and others), "On a Scheme for Calculating the Flux of Short-Wave Solar Radiation in the Atmosphere," *Meteorology and Hydrology*, No. 8, 1970.

30. Kim, V. F., "Certain Results of a Numerical Experiment on a Quasi-Geostrophic Model of Forecasting the Geopotential in a Hemisphere," *Izvestiia, Academy of Sciences, Physics of the Atmosphere and Ocean*, Vol. 4, No. 2, 1968.

31. Dmitrieva-Arrago, L. R., L. F. Koloskova, and L. S. Orlova, "Experience with the J. Smagorinsky Graph for Determining the Amount of Cloudiness," *Trudy, Main Geophysical Observatory*, Vol. 236, 1969.

32. Manabe, S., J. Smagorinsky, and R. F. Strickler, "Simulated Climatology of a General Circulation Model with a Hydrologic Cycle," *Monthly Weather Review*, Vol. 93, No. 12, 1965 (Russian translation appears in No. 1).

33. Kolsky, H. G., *A Study of the Physical, Mathematical and Computer Problems Behind Global Numerical Weather Calculations* [IBM report], TR 38.002, July 6, 1966.